Internet Appendix for "Bubbles and the Value of Innovation"[†]

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Abstract

This Internet Appendix provides additional information supporting the main text. Specifically, in Section A we detail the data construction. Section B discusses the assumptions of the model. Section C studies the asymptotics of large disagreement. Section D considers extensions of our framework. Section E provides additional empirical results.

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A Data Construction Details

Tech Distance. Bloom, Schankerman, and Van Reenen (2013) propose a variety of metrics for distance. We follow their construction. Proximity between two competing firms in the product market space is the correlation of the firms' distribution of sales across their industry segments. Technology proximity is analogously defined as the correlation of patent USPTO technology classes between firms, following earlier work by Jaffe (1986). The quantities of innovation by competitors and close innovators are the stocks of innovation weighted by these correlations. The stock of innovation is constructed using a perpetual-inventory approach. Both measures are extended using the Mahalanobis distance to allow for flexible weighting of the correlation between firms across different technology or product market classes. Further details on the measures of spillovers can be found in Bloom, Schankerman, and Van Reenen (2013).

Value of Spillovers. We obtain information on the quantity of competition spillovers (variable spillsic) as well as technological spillovers (variable spilltec) from the replication files in Bloom, Schankerman, and Van Reenen (2013). The exposure to spillovers from product market, *spillsic*, is defined as the correlation of the sales across two firms' Compustat segments. If we consider the vector of average sales share across each industry for a given firm i, $\mathbf{S}_i$, product market proximity between firm i and j is defined by the uncentered correlation: $\textsc{sic}_{ij} = \mathbf{S}_i\mathbf{S}_j' / \left(\sqrt{\mathbf{S}_i\mathbf{S}_i'}\sqrt{\mathbf{S}_j\mathbf{S}_j'}\right)$. The product market spillover is the average stock of R&D that is in the product market proximity of firm i:

$$spillsic_i = \sum_{j \neq i} \textsc{sic}_{ij} G_j,$$

where G_j is the stock of R&D for firm j. The exposure to knowledge spillovers is constructed the same way, where we define for firm i a vector of share of patents across technology classes from the USPTO as $\mathbf{T}_i$. The uncentered correlation of technology between firm i and j is: $\textsc{tech}_{ij} = \mathbf{T}_i\mathbf{T}_j' / \left(\sqrt{\mathbf{T}_i\mathbf{T}_i'}\sqrt{\mathbf{T}_j\mathbf{T}_j'}\right)$. The product market spillover is the average stock of R&D that is in the product market proximity of firm i:

$$spilltech_i = \sum_{j \neq i} \textsc{tech}_{ij} G_j.$$

To look at the effect of spillovers, we use sales item from Compustat funda file and Tobin's q (market-to-book ratio) from the CRSP-Compustat merged file.

Compustat Segments. We merge the Compustat funda file with the Compustat segments file. We estimate the number of segments with different industry codes. The Compustat segments file provides both a six- and a four-digit industry code, which gives two measures of the number of different types of industries within a public firm.

B Further Discussion of the Model

B.1 Microfoundations

The allocation of production slots is the key assumption to capture the notion of business-stealing. Indeed, firms do not internalize that they might take over the slot of another firm. The assumption of a fixed mass of production slots is especially plausible in industries that depend heavily on innovation. For instance, intellectual property law often provides exclusive use of a technology to its inventor.[1] We can interpret the fixed production slots as corresponding to a fixed number of M processes to produce the homogeneous good. The first firm to discover a process gets its exclusive use, and the speed of discovery is perfectly correlated with the productivity type a. Alternatively, we can assume that to produce, a firm needs one unit of an indivisible good that has not been discovered yet, and that only M of these exist in nature. Again firms with a higher type a find the ingredient faster.[2] Our assumption relies more generally on scarcity in the ability to produce that is not internalized by individual firms. We show in Internet Appendix Section D how our results hold for a wide range of models of business stealing. In particular, we show that the rigidity of a fixed number of active firms M is not necessary to our conclusions.

B.2 Calculation of Spillovers

Computing the spillovers in the model is challenging because the baseline model of Section 3 does not feature unexpected entry: the number of firms is deterministic in equilibrium. We overcome this challenge by introducing the possibility that some blueprints randomly fail to be implemented. Formally, after firm creators purchase blueprints but before they introduce these blueprints to public markets, a fraction of them disappear. If M_e blueprints are created, either they all succeed or a mass Δ fails, with probability $1-\varepsilon$ and ε, respectively. By focusing on the limit when ε and Δ go to zero, we trace out the equilibrium effect of the unanticipated entry of an atomistic firm on the total value of the economy.

Comparing the total market value of the economy across the two outcomes, we obtain the total effect of an extra firm on the value of the economy:

$$\lim_{\Delta \to 0} \frac{M_e V^{(n)}(M_e) - (M_e - \Delta) V^{(n)}(M_e - \Delta)}{\Delta} = V^{(n)}(M_e) + M_e V^{(n)\prime}(M_e) \tag{IA.1}$$

The outcome-based value of this measure is simply obtained by replacing n by 1 in this expression. The first term is the direct effect: the private value of a new

[1] Another motivation for the incompleteness is the difficulty of establishing markets for what has not been encountered yet. It is often the case that nobody owns something before it is discovered. For instance, how could we trade nuclear power before Henri Becquerel, Marie Curie, and Pierre Curie discovered radioactivity?

[2] Network goods or industries facing institutional constraints can face similar frictions in the allocations of productive positions that lead to a business-stealing effect—see Borjas and Doran (2012) for evidence in the context of scientific research.

firm $V^{(n)}(M_e)$. The second term is the change in the value of all other firms in response to the entry of the new firm. Each of the M_e firms in the economy sees its value decrease by $V^{(n)\prime}(M_e)$. Taking the ratio of the direct and the indirect effect, we obtain the two measures of spillovers in equations (12) and (13).

C Asymptotics

We now consider the limiting case of high speculation: $n \to \infty$. This case is extreme: the total quantity of entry goes to infinity. However, it is useful because it gives rise to sharp characterizations of the spillovers, allowing us to highlight the distinction between market-based and outcome-based spillovers with disagreement.

In our derivations, we will make use of the change of variable:

$$\begin{aligned} V^{(n)} &= \int_{\underline{a}}^{\infty} a^{\eta} \gamma n a^{-\gamma-1} \left(1 - a^{-\gamma}\right)^{n-1} da \\ &= \gamma n \underline{a}^{\eta-\gamma} \int_{1}^{\infty} t^{\eta-\gamma-1} \left(1 - \underline{a}^{-\gamma} t^{-\gamma}\right)^{n-1} dt. \end{aligned} \tag{IA.2}$$

Proposition C.1. *In the high-disagreement limit ($n \to \infty$), the market-based spillover converges to a finite limit that depends on the sign of $\gamma\theta - \eta$:*

- *If $\gamma\theta > \eta$, the market-based spillover vanishes:* $\lim_{n\to\infty} spill_{mkt}(n) = 0$
- *If $\gamma\theta < \eta$, then τ converges to the spillover in the agreement case ($n = 1$):* $\lim_{n\to\infty} spill_{mkt}(n) = -\frac{\gamma-\eta}{\gamma}$
- *In the knife-edge case of $\gamma\theta = \eta$,* $\lim_{n\to\infty} spill_{mkt}(n) = \check{spill}_{mkt} > -\frac{\gamma-\eta}{\gamma}$*, where $\check{spill}_{mkt}$ is defined in Appendix equation* (IA.4)*.*

Figure IA.1 illustrates these cases.

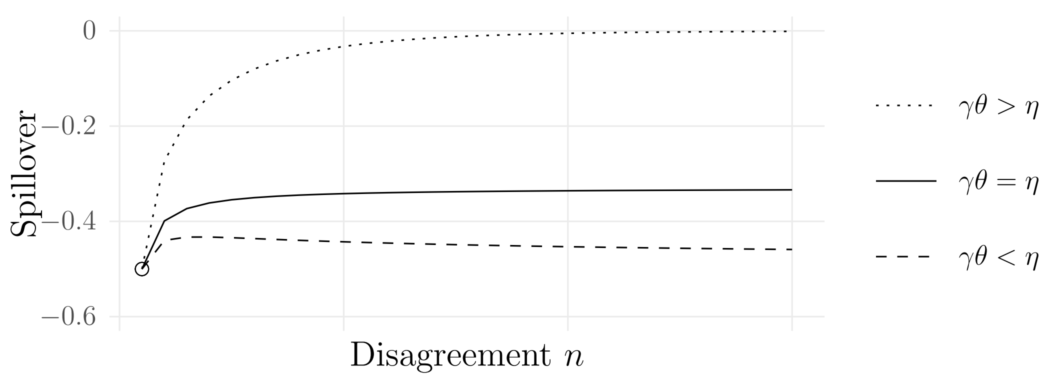

Figure IA.1
Market-based spillover with increasing disagreement.

We use the following two lemmas to prove Proposition C.1.

Lemma C.1. *If $\theta \geq 0$, then as disagreement increases ($n \to \infty$), the mass of entrants also increases and goes to infinity:* $\lim_{n\to\infty} M_e = \infty$*.*

Proof. We define $\underline{a}_n = (M_e/M)^{1/\gamma}$, where M_e now depends on n, and show that $\underline{a}_n \to \infty$. Equations (11) and (IA.2) imply an implicit definition of the sequence $\underline{a}_n$:

$$f_e \underline{a}_n^{\gamma(\theta+1)-\eta} = \gamma n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}_n^{-\gamma} t^{-\gamma}\right)^{n-1} dt.$$

Suppose $\underline{a}_n$ has a finite limit that is strictly larger than zero, i.e., $\underline{a}_\infty > 0$.[3] Then there exists N large enough such that $\forall n > N$, $\underline{a}_n > A = \underline{a}_\infty - \epsilon > 0$. We obtain a lower bound for the right-hand side of the implicit equation above:

$$\begin{aligned} I_n &= \gamma n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}^{-\gamma} t^{-\gamma}\right)^{n-1} dt \\ &> \gamma n \int_1^\infty t^{\eta-\gamma-1} \left(1 - A^{-\gamma} t^{-\gamma}\right)^{n-1} dt. \end{aligned}$$

Consider an arbitrary threshold T_n that depends on n and satisfies:

$$\begin{aligned} I_n &> \gamma n \int_{T_n}^\infty t^{\eta-\gamma-1} \left(1 - A^{-\gamma} t^{-\gamma}\right)^{n-1} dt \\ &> \gamma n \left(1 - A^{-\gamma} T_n^{-\gamma}\right)^{n-1} \int_{T_n}^\infty t^{\eta-\gamma-1} dt \\ &= \frac{\gamma}{\gamma - \eta} \cdot n \cdot T_n^{\eta-\gamma} \left(1 - A^{-\gamma} T_n^{-\gamma}\right)^{n-1}. \end{aligned}$$

Choose the threshold $T_n = n^{1/\gamma}$. The bound becomes:

$$\begin{aligned} I_n &> \frac{\gamma}{\gamma - \eta} \cdot n^{\frac{\eta}{\gamma}} \exp\left(-(n-1)\log(1 - A^{-\gamma} n^{-1})\right) \\ &> \frac{\gamma}{\gamma - \eta} \cdot n^{\frac{\eta}{\gamma}} \exp\left(-(n-1) A^{-\gamma} n^{-1} + \mathcal{O}(n^{-1})\right). \end{aligned}$$

Since $\gamma(\theta+1) - \eta \geq \gamma - \eta > 0$, this implies $I_n \to \infty$, contradicting $\underline{a}_\infty < \infty$. ∎

Lemma C.2 (Asymptotics for firm creation)**.** *In the high-disagreement limit ($n \to \infty$), we have the following asymptotics for the mass of firms created, M_e:*

- *If $\gamma\theta < \eta$, then* $M_e/M = \left(\frac{1}{f_e}\frac{\gamma}{\gamma-\eta} \cdot n\right)^{\frac{\gamma}{\gamma(\theta+1)-\eta}}$.
- *If $\gamma\theta = \eta$, then* $\lim_{n\to\infty} M_e/M = \alpha_\infty n$*, where α_∞ is a constant defined below.*

Proof. Substituting $\underline{a}$ into (11), we have:

$$\begin{aligned} f_e &= \gamma \underline{a}^{\eta-\gamma(\theta+1)} n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}^{-\gamma} t^{-\gamma}\right)^{n-1} dt \\ &\simeq \gamma \underline{a}^{\eta-\gamma(\theta+1)} n \int_1^\infty t^{\eta-\gamma-1} \exp\left(-(n-1)\underline{a}^{-\gamma} t^{-\gamma}\right) dt, \end{aligned}$$

[3] Since the mass of firms producing cannot be higher than the mass of firms created, $\underline{a}_n \geq 1$.

where we have used the fact that $\underline{a} \to \infty$ from Lemma C.2, and $\log(1-x) = -x + \mathcal{O}(x^2)$. To find a solution, we guess the asymptotics of $\underline{a}(n)$. We rewrite $\underline{a} = \alpha(n) n^{1/(\gamma(1+\theta)-\eta)}$ and show that $\alpha(n)$ converges to a finite limit α. The above equation becomes:

$$f_e = \gamma \alpha(n)^{\eta-\gamma(\theta+1)} \int_1^\infty t^{\eta-\gamma-1} \exp\left(-\alpha(n)^{-\gamma} \frac{n-1}{n^{\frac{\gamma}{\gamma(1+\theta)-\eta}}} t^{-\gamma}\right) dt.$$

Suppose $\gamma\theta < \eta$. Then the exponential term converges to zero and we have:

$$f_e = \gamma \alpha^{\eta-\gamma(\theta+1)} \int_1^\infty t^{\eta-\gamma-1} dt = \alpha^{\eta-\gamma(\theta+1)} \frac{\gamma}{\gamma-\eta},$$

such that we have the following asymptotics for firm entry:

$$\frac{M_e}{M} = \left(\frac{1}{f_e} \frac{\gamma}{\gamma-\eta} \cdot n\right)^{\frac{\gamma}{\gamma(\theta+1)-\eta}}. \tag{IA.3}$$

Suppose $\gamma\theta = \eta$. Then $\underline{a}$ is defined by:

$$f_e = \gamma \underline{a}_n^{-\gamma} n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}^{-\gamma} t^{-\gamma}\right)^{n-1} dt.$$

Since $\underline{a} = (M_e/M)^{1/\gamma}$, it is sufficient to guess and verify that $\underline{a}_n = \alpha(n)^{-1/\gamma} n^{1/\gamma}$, and $\alpha(n)$ has a finite limit α_∞ defined by:

$$\begin{aligned} f_e &= \gamma \alpha(n) \int_1^\infty t^{\eta-\gamma-1} \exp\left(-(n-1)(\alpha(n) n^{-1} + \mathcal{O}(\alpha(n)^2 n^{-2})) t^{-\gamma}\right) dt \\ &\xrightarrow[n\to\infty]{} \gamma \alpha_\infty \int_1^\infty t^{\eta-\gamma-1} e^{-\alpha_\infty t^{-\gamma}} dt, \end{aligned}$$

where we take the limit when $n \to \infty$. The outcome-based spillover implies:

$$f_e > \gamma \alpha_\infty e^{-\alpha_\infty} \int_1^\infty t^{\eta-\gamma-1} dt > \alpha_\infty e^{-\alpha_\infty} \frac{\gamma}{\gamma-\eta},$$

and thus

$$\frac{\alpha_\infty e^{-\alpha_\infty}}{f_e} < \frac{\gamma-\eta}{\gamma}, \tag{IA.4}$$

which implies a finite bound on α_∞. ∎

Using the asymptotics derived in Lemma C.2, we now prove Proposition C.1.

Proof. **(Proposition C.1)** Suppose $\gamma\theta < \eta$. Substitute the asymptotics derived in equation (IA.3) into the formula for the market-based spillover:

$$spill_{mkt}(n; M_e) = -\frac{\frac{M}{M_e}\left(\frac{M_e}{M}\right)^{\frac{\eta}{\gamma}} n \left(1 - \frac{M}{M_e}\right)^{n-1}}{f_e \left(\frac{M_e}{M}\right)^{\theta}} \tag{IA.5}$$

$$\simeq -\frac{1}{f_e} \cdot f_e \frac{\gamma - \eta}{\gamma}\frac{1}{n} \cdot n \left(1 - \frac{M}{M_e}\right)^{n-1} \to -\frac{\gamma - \eta}{\gamma}, \tag{IA.6}$$

where we have used the fact that $(1 - M/M_e)^{n-1} \to 1$.[4] The market-based spillover therefore converges to the outcome-based spillover in this case.

Now suppose $\gamma\theta > \eta$. We write the market-based spillover directly:

$$spill_{mkt}(n; M_e) = -\frac{n\underline{a}^{\eta-\gamma}(1 - \underline{a}^{-\gamma})^{n-1}}{f_e \underline{a}^{\gamma\theta}}.$$

First suppose $\underline{a} \to \infty$. We rewrite the competitive equilibrium condition of equation (11):

$$n\underline{a}^{-\gamma} = \frac{f_e \underline{a}^{\gamma\theta-\eta}}{\gamma \int_1^{\infty} t^{\eta-\gamma-1}(1 - \underline{a}^{-\gamma}t^{-\gamma})^{n-1} dt}.$$

The denominator is bounded from above by $\gamma \int_1^{\infty} t^{\eta-\gamma-1} dt$, which implies $n\underline{a} \to \infty$. Using a first-order approximation, we have:

$$(1 - \underline{a}^{-\gamma})^{n-1} \simeq \exp\left(-n\underline{a}^{-\gamma}\right).$$

Therefore, the market-based spillover in the limit is:

$$spill_{mkt}(n) \simeq -\frac{n\underline{a}^{-\gamma}\exp\left(-n\underline{a}^{-\gamma}\right)}{f_e \underline{a}^{\gamma\theta-\eta}} \to 0, \tag{IA.7}$$

since the numerator goes to zero and the denominator goes to infinity. Suppose instead that $\underline{a}$ has a finite limit. We obtain the expression for $spill_{mkt}$:

$$spill_{mkt}(n) = -\frac{n(1 - \underline{a}^{-\gamma})^{n-1}}{f_e \underline{a}^{\gamma(1+\theta)-\eta}} = -\frac{n\exp\left((n-1)\log(1 - \underline{a}^{-\gamma})\right)}{f_e \underline{a}^{\gamma(1+\theta)-\eta}} \to 0, \tag{IA.8}$$

since the denominator has a finite limit and the numerator goes to 0.

Lastly, consider the case where $\gamma\theta = \eta$. The spillover expression simplifies to:

$$spill_{mkt}(n) = -\frac{1}{f_e} \cdot n\underline{a}^{-\gamma}\left(1 - \underline{a}^{-\gamma}\right)^{n-1}.$$

[4] This follows from $(1 - M/M_e)^{n-1} = \exp\left[-(n-1)\log(M_e/M)\right]$ and using the asymptotics derived above for $\gamma\theta < \eta$: $(1 - M/M_e)^{n-1} = \exp\left[-(n-1)\left(f_e^{-1}\frac{\gamma}{\gamma-\eta}n\right)^{-\frac{\gamma}{\gamma(1+\theta)-\eta}}\right] \to 1$.

Using Lemma C.2 and the result that $\underline{a}^{-\gamma} = \alpha(n)/n$, and $\alpha(n) \to \alpha_\infty$, we have:

$$spill_{mkt}(n) \simeq -\frac{1}{f_e}\alpha(n)\exp\left(-(n-1)\alpha(n)/n\right) \tag{IA.9}$$

$$\simeq -\frac{1}{f_e}\alpha(n)\exp\left(-\alpha(n)\right) \to -\frac{1}{f_e}\alpha_\infty e^{-\alpha_\infty}. \tag{IA.10}$$

Moreover, using Lemma C.2, this also proves that in the limit, the magnitude of the market-based spillover $|spill_{mkt}|$ is less than that of the outcome-based spillover $|spill_{out}| = (\gamma - \eta)/\gamma$. ∎

D Extensions

We now consider several extensions of the model:

- Appendix D.1 allows for a more general form of the business-stealing effect. Instead of having a single cutoff below which firms do not produce, we allow firms to produce with a probability that decreases with their rank in the economy.
- Appendix D.2 breaks the result in our baseline that marginal firm earns positive profits. Instead, firms compete for the production slots, which yields a zero-cutoff-profit condition for the marginal firm.
- Appendix D.3 considers three ways endogenize the profit function of equation (4): labor with decreasing returns to scale, Dixit-Stiglitz aggregate demand, and knowledge spillovers. These models introduce additional sources of spillovers.
- Appendix D.4 considers situations with a variable number of participating firms.
- Appendix D.5 introduces participation costs to provide an alternative microfoundation for a zero-cutoff-profit condition. We derive results for both the baseline model and the model of Appendix D.3 as in Melitz (2003).

These extensions serve three purposes. First, they show the robustness of our results. Next, they show implications of our model of disagreement for spillovers beyond the business-stealing effect that we focus on in the main text. Finally, they provide further examples of the tractability of our framework.

D.1 Generalizing the Business-Stealing Effect

Our results are robust to more general functions for the business-stealing effect. In particular, suppose the expected profit of a firm with productivity a is:

$$\pi(a) = a^\eta \delta\left(r(a, M_e)\right),$$

where $r(a, M_e) \equiv (1 - F(a))\, M_e$ is the ranking of the firm, or the mass of firms with productivity greater than a. The function δ is the probability of producing conditional on a firm's ranking r. The main text focused on the special case of $\delta(r) = \mathbf{1}\{r \leq M\}$.

D.1.1 Outcome-Based Spillover

Lemma D.1. *The outcome-based spillover does not depend on the level of entry:*

$$spill_{out} = -\frac{\gamma - \eta}{\gamma}. \tag{IA.11}$$

Proof. Under agreement, $n = 1$, and we can derive an exact solution for the mass of firms entering in equilibrium M_e. Integrating by parts, the value of a firm is:

$$\begin{aligned} V^{(1)}(M_e) &= \int_1^\infty \gamma x^{\eta-\gamma-1} \delta(M_e x^{-\gamma}) dx \\ &= \frac{\gamma}{\gamma - \eta} \left[\delta(M_e) - \gamma M_e \int_1^\infty x^{\eta-2\gamma-1} \delta'(M_e x^{-\gamma}) dx \right]. \end{aligned}$$

In addition, we have:

$$M_e \frac{dV^{(1)}}{dM_e} = \gamma M_e \int_1^\infty x^{\eta-2\gamma-1} \delta'(M_e x^{-\gamma}) dx.$$

Recalling that $spill_{out} = M_e \frac{dV^{(1)}}{dM_e} / V^{(1)}$, we have the desired formula (IA.11). ∎

D.1.2 Disagreement Asymptotics with Multiple Cutoffs

We now consider the generalization of δ to allow for multiple cutoffs. In particular, suppose we have cutoffs $\underline{a}_1 < ... < \underline{a}_K$, with $\underline{a}_k \equiv F^{-1}(1 - \frac{M_k}{M_e})$, and constants $\Delta_1, ..., \Delta_K$ so that

$$\delta(r) = \sum_{k=1}^{K} \Delta_k \mathbf{1}\left\{ r \leq M_k \right\}. \tag{IA.12}$$

Notice that this imples that $V^{(n)} = \sum_{k=1}^{K} \Delta_k V_k^{(n)}$, where $V_k^{(n)} \equiv \int_{\underline{a}_k}^{\infty} a^\eta dF_n(a)$, and

$$-M_e \frac{dV^{(n)}}{dM_e} = -\sum_{k=1}^{K} \left(\Delta_k \frac{M_k}{M_e} \cdot a^\eta \cdot \frac{F_n'}{F'}(\underline{a}_k) \right).$$

For convenience, we normalize the cost of producing blueprints so that $W(b) = f_e b^{\theta+1} M_K^{-\theta}/(\theta + 1)$.

Lemma D.2. *Holding M_e constant, the outcome-based spillover is larger than the market-based spillover.*

Proof. Apply the proof for Proposition 1 for each k. ∎

Theorem D.3 (Asymptotics for the market-based spillover with multiple cutoffs)**.** *With business-stealing of the form (IA.12), in the high-disagreement limit ($n \to \infty$), the market-based spillover converges to a finite limit.*

- *If $\gamma\theta < \eta$, then $spill_{mkt}(n) \to -(\gamma - \eta)/\gamma$.*
- *If $\gamma\theta > \eta$, then $spill_{mkt}(n) \to 0$.*

- *If $\gamma\theta = \eta$, then $spill_{mkt}(n) \to -\frac{1}{f_e}\sum_{k=1}^{K}\left[\Delta_k \left(M_K/M_k\right)^{(\eta-\gamma)/\gamma} \alpha_\infty \exp\left(-\alpha_\infty M_k/M_K\right)\right]$.*

Proof. Suppose $\gamma\theta < \eta$. Conjecturing as before that we can write $\underline{a}_K = \alpha(n)n^{1/(\gamma(1+\theta)-\eta)}$ yields

$$f_e \simeq \sum_{k=1}^{K} \Delta_k \left(\frac{M_k}{M_K}\right)^{\frac{\gamma-\eta}{\gamma}} \alpha^{\eta-\gamma(\theta+1)} \frac{\gamma}{\gamma-\eta}.$$

Therefore, we have the asymptotics for firm entry:

$$\frac{M_e}{M_K} = \left[\frac{1}{f_e}\sum_{k=1}^{K} \Delta_k \left(\frac{M_k}{M_K}\right)^{\frac{\gamma-\eta}{\gamma}} \frac{\gamma}{\gamma-\eta}\cdot n\right]^{\frac{\gamma}{\gamma(\theta+1)-\eta}}.$$

Substituting this into the formula for the market-based spillover, we have

$$spill_{mkt}(n) = -\frac{\sum_{k=1}^{K} \Delta_k \left(\frac{M_k}{M_e}\right)^{\frac{\gamma-\eta}{\gamma}} n\left(1-\frac{M_k}{M_e}\right)^{n-1}}{f_e\left(\frac{M_e}{M_K}\right)^{\theta}} \to -\frac{\gamma-\eta}{\gamma} \qquad \text{(IA.13)}$$

as desired.

Now suppose $\gamma\theta > \eta$. Then we have

$$\begin{aligned} spill_{mkt}(n) &= -\frac{n\sum_{k=1}^{K} \underline{a}_k^{\eta-\gamma}\left(\underline{a}_k^{-\gamma}\right)^{n-1}}{f_e \underline{a}_K^{\gamma\theta}} \\ &= -\frac{n\sum_{k=1}^{K}\left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \underline{a}_K^{-\gamma}\left(1-\frac{M_k}{M_K}\underline{a}_K^{-\gamma}\right)^{n-1}}{f_e\underline{a}_K^{\gamma\theta-\eta}} \end{aligned}$$

Suppose $\underline{a}_K \to \infty$. Then we can write the first-order condition for firm creation as:

$$\frac{f_e \underline{a}_K^{\gamma\theta-\eta}}{n\underline{a}_K^{-\gamma}} = \sum_{k=1}^{K} \Delta_k \left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \gamma \int_1^{\infty} t^{\eta-\gamma-1}\left(1-\underline{a}_k^{-\gamma}t^{-\gamma}\right)^{n-1} dt.$$

Since the integral on the right-hand side is bounded from above by $\int_1^\infty t^{\eta-\gamma-1}dt$, we have that $n\underline{a}_K^{-\gamma} \to \infty$, which implies that we can use a similar approximation to the proof of Proposition C.1 to show that:

$$spill_{mkt}(n) \simeq -\frac{n\sum_{k=1}^{K}\left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \underline{a}_K^{-\gamma} \exp\left(-n\frac{M_k}{M_K}\underline{a}_K^{-\gamma}\right)}{f_e\underline{a}_K^{\gamma\theta-\eta}} \to 0. \qquad \text{(IA.14)}$$

Finally, suppose $\gamma\theta = \eta$. We then have

$$f_e = \sum_{k=1}^{K} \Delta_k \gamma n \left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \underline{a}_K^{-\gamma} \int_1^{\infty} t^{\eta-\gamma-1}\left(1-\frac{M_k}{M_K}\underline{a}_K^{-\gamma}t^{-\gamma}\right).$$

As before, we conjecture that $\underline{a}_K = \alpha(n)^{-1/\gamma} n^{1/\gamma}$. Then

$$\begin{aligned} f_e &\simeq \sum_{k=1}^{K} \Delta_k \left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \gamma\alpha(n) \int_1^\infty t^{\eta-\gamma-1} \exp\left[-(n-1)\alpha(n)n^{-1}\frac{M_k}{M_K}t^{-\gamma}\right] dt \\ &\to \sum_{k=1}^{K} \Delta_k \left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \gamma\alpha_\infty \int_1^\infty t^{\eta-\gamma-1} \exp\left[-\frac{M_k}{M_K}\alpha_\infty t^{-\gamma}\right] dt. \end{aligned}$$

By analogous reasoning to the proof in Proposition C.1, we can obtain a finite bound on α_∞. We therefore have the market-based spillover:

$$\begin{aligned} spill_{mkt}(n) &= -\frac{1}{f_e} \sum_{k=1}^{K} \left[\Delta_k n \left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \underline{a}_K^{-\gamma} \left(1 - \frac{M_k}{M_K}\underline{a}_K^{-\gamma}\right)\right] \\ &\to -\frac{1}{f_e} \sum_{k=1}^{K} \left[\Delta_k \left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \alpha_\infty \left(1 - \frac{M_k}{M_K}\alpha_\infty\right)\right] \end{aligned} \quad \text{(IA.15)}$$

as desired. ■

D.1.3 Disagreement Asymptotics with Continuous Business-Stealing

We now consider a continuous function for the business-stealing effect

$$\delta(r) = \begin{cases} 1 & \text{if } r < 1 \\ r^{-\zeta} & \text{if } r \geq 1 \end{cases} \quad \text{(IA.16)}$$

so that ζ parameterizes the business-stealing effect for low-productivity firms with $r \geq 1$. Larger ζ implies that low-productivity firms have a lower probability of producing, with $\zeta = 0$ corresponding to the case with no business-stealing effect. With $\zeta \to \infty$, this converges to the benchmark step function business-stealing effect with $M = 1$.

We now normalize the cost of producing blueprints, so that $W(b) = f_e b^{\theta+1}/(\theta+1)$, and define $\underline{a} \equiv M_e^{1/\gamma}$ to be the cutoff above which $\delta(a, M_e) = 1$, i.e., firms produce with probability one.

It will be convenient to consider the decomposition $V^{(n)} = V_L^{(n)} + V_U^{(n)}$, where

$$\begin{aligned} V_L^{(n)} &\equiv \gamma n M_e^{-\zeta} \int_1^{\underline{a}} x^{\eta-\gamma(1-\zeta)-1}(1-x^{-\gamma})^{n-1} dx \\ V_U^{(n)} &\equiv \gamma n \int_{\underline{a}}^\infty x^{\eta-\gamma-1}(1-x^{-\gamma})^{n-1} dx \end{aligned}$$

capture the expected profit conditional on having productivity below and above

$\underline{a}$ respectively. We can write

$$V_L^{(n)} = \gamma n \underline{a}^{\eta-\gamma(\theta+1)} \int_{\underline{a}^{-1}}^{1} t^{\eta-\gamma(1-\zeta)-1}(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1}dt \tag{IA.17}$$

$$V_U^{(n)} = \gamma n \underline{a}^{\eta-\gamma(\theta+1)} \int_{1}^{\underline{a}^{-1}} t^{\eta-\gamma-1}(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1}dt. \tag{IA.18}$$

Moreover, since

$$\begin{aligned}\frac{dV_L^{(n)}}{dM_e} &= -\zeta M_e^{-1}V_L^{(n)} + \gamma n M_e^{-\zeta} M_e^{\frac{\eta-\gamma(1-\zeta)-1}{\gamma}}\left(1-M_e^{-1}\right)^{n-1} \\ &= -\zeta M_e^{-1}V_L^{(n)} - \frac{dV_U^{(n)}}{dM_e},\end{aligned}$$

we have that

$$-M_e\frac{dV^{(n)}}{dM_e} = \zeta V_L^{(n)}. \tag{IA.19}$$

Theorem D.4 (Asymptotics for the market-based spillover with continuous business-stealing)**.** *Suppose we have business stealing of the form (IA.16) and* $\zeta > \frac{\gamma-\eta}{\gamma}$*. In the high-disagreement limit* ($n\to\infty$)*, the market-based spillover converges to a finite limit.*

- *If* $\gamma\theta < \eta$*, then* $spill_{mkt}(n) \to -(\gamma-\eta)/\gamma$*.*
- *If* $\gamma\theta \geq \eta$*, then* $spill_{mkt}(n) \to 0$*.*

Proof. Suppose $\gamma\theta<\eta$. Conjecture that $\underline{a} = \alpha(n)n^{1/(\gamma(1+\theta)-\eta)}$. We have from the proof of Proposition C.1 that $V_U^{(n)} \to \alpha^{\eta-\gamma}\frac{\gamma}{\gamma-\eta}$. In addition, we have from (IA.17) that $V_U^{(n)} \to \alpha^{\eta-\gamma}\frac{\gamma}{\eta-\gamma(1-\zeta)}$. Therefore, we have

$$f_e = \left(\frac{\gamma}{\gamma-\eta} - \frac{\gamma}{\gamma(1-\zeta)-\eta}\right)\alpha^{\eta-\gamma(\theta+1)}, \tag{IA.20}$$

which verifies the conjecture. We thus have the asymptotic market-based spillover

$$spill_{mkt}(n) = -\zeta\frac{V_L^{(n)}}{V^{(n)}} \to -\frac{\gamma-\eta}{\gamma} \tag{IA.21}$$

as desired.

Suppose $\gamma\theta>\eta$. Suppose $\underline{a}\to\infty$. Then we can rewrite equation (11) as:

$$n\underline{a}^{-\gamma} = \frac{f_e\underline{a}^{\gamma\theta-\eta}/\gamma}{\int_{\underline{a}^{-1}}^{1} t^{\eta-\gamma(1-\zeta)-1}(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1}dt + \int_1^\infty t^{\eta-\gamma-1}(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1}dt}. \tag{IA.22}$$

The two terms in the denominator of the right-hand side are bounded from above by $\int_0^1 t^{\eta-\gamma(1-\zeta)-1}dt$ and $\int_1^\infty t^{\eta-\gamma-1}dt$, respectively, which implies that $n\underline{a}^{-\gamma}\to$

∞. Using the approximation $(1-\underline{a}^{-\gamma})^{n-1} \simeq \exp(-n\underline{a}^{-\gamma})$, we have that

$$spill_{mkt}(n) = -\frac{\zeta}{f_e \underline{a}^{\gamma\theta}} V_L^{(n)} \to 0. \tag{IA.23}$$

If $\underline{a}$ has a finite limit, we can show that $V_L^{(n)} \to 0$ since $n(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1} \to 0$ for $t \in (\underline{a}^{-1}, 1)$, which implies that $spill_{mkt}(n) \to 0$ as well.

Suppose $\gamma\theta = \eta$. Using an analogous proof to Lemma C.2, we can show that $\underline{a}_n = \alpha(n)^{-1/\gamma} n^{1/\gamma}$, where $\alpha(n)$ has a finite limit α_∞. Since we can bound $V_L^{(n)}$ from above by

$$\begin{aligned} V_L^{(n)} &= \gamma\alpha(n) \int_{\underline{a}^{-1}}^{1} t^{\eta-\gamma(1-\zeta)-1}(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1} dt \\ &\leq \gamma\alpha(n) \int_{\underline{a}^{-1}}^{1} t^{\eta-\gamma(1-\zeta)-1} dt \to \frac{\gamma\alpha_\infty}{\eta-\gamma(1-\zeta)}, \end{aligned}$$

we have that $spill_{mkt}(n) = -\frac{\zeta V_L^{(n)}}{f_e \underline{a}^{\gamma\theta}} \to 0$. ∎

D.2 Results with a Zero-Cutoff-Profit Condition

Our results are robust to an extension in which the marginal firm earns zero profits. Our baseline model specifies that the M most productive firms will be allowed to produce, which allows for tractability but results in the marginal firm earning positive profits, $\pi(\underline{a}) > 0$. We now augment the model with an intermediate stage where firms, after entering the market, compete to be among one of the M firms producing. The competition stage ensures that the business-stealing externality remains. We keep the belief and production structure of the model intact and show that the main features of the spillovers remain unchanged despite the introduction of a zero-cutoff-profit (ZCP) condition for the marginal firm.

In the new intermediate decision stage, firms can use some of their production as advertisement to reach consumers, a deadweight loss. Only the M firms that spend the most on advertising produce in equilibrium. Formally, each firm chooses how much of its production to use on advertisement, $h_i \leq \pi(a_i)$. In doing so, firms take as given the equilibrium level $\underline{h}$ of advertising necessary to attract consumers. Their profit function is therefore $\pi(a_i)\mathbf{1}\{h_i \geq \underline{h}\} - h_i$. The optimal advertisement choice is $h_i = \underline{h}$ if $\pi(a_i) \geq \underline{h}$ and 0 otherwise. The equilibrium value of $\underline{h}$ is such that exactly M firms choose to spend on advertisement. Keeping the definition of the production cutoff $\underline{a}$ from earlier, this implies

$$\underline{h} = \pi(\underline{a}). \tag{IA.24}$$

Firms must spend the profits of the marginal firm to be able to produce, resulting in zero profits for the marginal firm.

D.2.1 General Derivations

Firm value in this model is modified to account for the cost of advertisement:

$$\widetilde{V}^{(n)}(M_e) = \int_{\underline{a}}^{\infty} (\pi(a) - \pi(\underline{a}))\, dF^n(a). \tag{IA.25}$$

We can define the corresponding integral $\widetilde{\mathcal{I}}_n$. With this new definition of firm value, the remainder of the competitive equilibrium and the planner problem are unchanged. In particular, the market-based spillover is $spill_{mkt} = \mathcal{E}_{\widetilde{\mathcal{I}}_n}$.

Decompose firms' valuations into the revenue (from (IA.25)) and advertising cost components:

$$V^{(n)}(M_e) = \int_{F^{-1}\left(1-\frac{M}{M_e}\right)}^{\infty} \pi(a)dF_n(a) - \left(\frac{M}{M_e}\right)^{1-\frac{\eta}{\gamma}} \cdot n\left(1-\frac{M}{M_e}\right)^{n-1}. \tag{IA.26}$$

The first derivative of $V^{(n)}$ is:

$$-M_e \cdot \frac{dV^{(n)}(M_e)}{dM_e} = \frac{\eta}{\gamma}\left(\frac{M_e}{M}\right)^{\frac{\eta}{\gamma}} \cdot \left[1-\left(1-\frac{M}{M_e}\right)^{n}\right].$$

Using the free-entry condition, $V^{(n)}(M_e) = W'(M_e)$, we have following formula for the market-based spillover:

$$spill_{mkt}(n; M_e) = \frac{\eta}{\gamma} \cdot \frac{1}{f_e} \cdot \left(\frac{M_e}{M}\right)^{\frac{\eta}{\gamma}-\theta} \cdot \left[1-\left(1-\frac{M}{M_e}\right)^{n}\right]. \tag{IA.27}$$

D.2.2 Spillovers and Firm Entry

Lemma D.5. *In the model with a ZCP condition, the outcome-based spillover is:*

$$spill_{out} = -\frac{\gamma-\eta}{\gamma}.$$

Proof. The free-entry condition with $n = 1$ gives us:

$$\left(\frac{M_e}{M}\right)^{\frac{\gamma(1+\theta)-\eta}{\gamma}} = \frac{1}{f_e} \cdot \frac{\eta}{\gamma-\eta}.$$

Given the derivation of the spillover in (IA.27), we have:

$$spill_{out}(M_e) = -\frac{\eta}{\gamma} \cdot \frac{1}{f_e} \cdot \left(\frac{M_e}{M}\right)^{\frac{\eta}{\gamma}-\theta} \cdot \frac{M}{M_e} = -\frac{\gamma-\eta}{\gamma}, \tag{IA.28}$$

where we have used our equilibrium solution for M_e/M. ∎

Lemma D.6. *In the high-disagreement limit ($n \to \infty$), the mass of entrants also increases and goes to infinity:* $\lim_{n\to\infty} M_e = \infty$.

Proof. We adapt the proof from Lemma C.1, again defining the sequence $\underline{a}_n = (M_e/M)^{1/\gamma}$ and showing that $\underline{a}_n \to \infty$. Equation (IA.26) implies the implicit definition of the sequence $(\underline{a}_n)_n$ in this case:

$$f_e \underline{a}_n^{\gamma(\theta+1)-\eta} = \gamma n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}_n^{-\gamma} t^{-\gamma}\right)^{n-1} dt - n(1 - \underline{a}_n^{-\gamma})^{n-1}.$$

Assume that $\underline{a}_n$ has a finite limit that is strictly larger than zero, $\underline{a}_\infty > 0$. Then there exists N large enough such that $\forall n > N, \underline{a}_n > A = \underline{a}_\infty - \epsilon > 0$. For any arbitrary threshold T_n, we have

$$I_n > n \left[\frac{\gamma}{\gamma - \eta} \cdot T_n^{\eta-\gamma} \left(1 - A^{-\gamma} T_n^{-\gamma}\right)^{n-1} - 1\right].$$

As in Lemma C.1, we conclude by considering the threshold $T_n = n^{1/\gamma}$. ∎

Lemma D.7 (Asymptotics for firm creation)**.** *In the high-disagreement limit ($n \to \infty$), we have the following asymptotics for the mass of firms created M_e:*

- *If $\gamma\theta < \eta$, then $\lim_{n\to\infty} M_e/M = \alpha_\infty^\gamma n^{\frac{\gamma}{\gamma(1+\theta)-\eta}}$.*
- *If $\gamma\theta = \eta$, then $\lim_{n\to\infty} M_e/M = \alpha_\infty^{-1} n$.*

In each case, α_∞ is a constant defined below.

Proof. We adapt the proof from Lemma C.2. Starting from equation (11), and using $\underline{a}$:

$$f_e \simeq \gamma \underline{a}^{\eta-\gamma(\theta+1)} n \int_1^\infty \left(t^\eta - 1\right) t^{-\gamma-1} \exp\left(-(n-1)\underline{a}^{-\gamma} t^{-\gamma}\right) dt,$$

where we have used the fact that $\underline{a} \to \infty$ and $\log(1-x) = -x + \mathcal{O}(x^2)$. To find a solution, we guess that asymptotically $\underline{a} \simeq \alpha(n) n^{\frac{1}{\gamma(1+\theta)-\eta}}$ and show that $\alpha(n)$ converges to a finite limit α. The first-order condition becomes

$$f_e \simeq \gamma \alpha(n)^{\eta-\gamma(\theta+1)} \int_1^\infty \left(t^\eta - 1\right) t^{-\gamma-1} \exp\left(-\alpha(n)^{-\gamma} \frac{n-1}{n^{\frac{\gamma}{\gamma(1+\theta)-\eta}}} t^{-\gamma}\right) dt.$$

If $\gamma\theta < \eta$, then the exponential term converges to zero and we have:

$$\alpha_\infty = \left(\frac{1}{f_e} \cdot \frac{\eta}{\gamma - \eta}\right)^{\frac{1}{\gamma(1+\theta)-\eta}}. \tag{IA.29}$$

If $\gamma\theta = \eta$, then $\underline{a}$ is defined by:

$$f_e = \gamma \underline{a}_n^{-\gamma} n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}^{-\gamma} t^{-\gamma}\right)^{n-1} dt.$$

We guess and verify that $\underline{a}_n = \alpha(n)^{-1/\gamma} n^{1/\gamma}$, and $\alpha(n)$ has a finite limit α_∞:

$$f_e = \gamma \alpha_\infty \int_1^\infty \left(t^\eta - 1\right) t^{-\gamma-1} e^{-\alpha_\infty t^{-\gamma}} dt,$$

where we took the limit when $n \to \infty$. Moreover, we are able to bound the magnitude of the market-based spillover above using a bound on α_∞:

$$\frac{\alpha_\infty e^{-\alpha_\infty}}{f_e} < \frac{\gamma - \eta}{\eta}, \tag{IA.30}$$

which verifies that α_∞ is finite. ∎

We now show that, despite the presence of the ZCP, Proposition C.1 holds.

Theorem D.8. *In the high-disagreement limit ($n \to \infty$), the market-based spillover converges to a finite limit.*

- *If $\gamma\theta < \eta$, then $spill_{mkt}(n) \to -(\gamma - \eta)/\gamma$.*
- *If $\gamma\theta > \eta$, then $spill_{mkt}(n) \to 0$.*
- *If $\gamma\theta = \eta$, then $spill_{mkt}(n) \to -\frac{\eta}{\gamma}\frac{1}{f_e}e^{-\alpha_\infty}$.*

Proof. If $\gamma\theta > \eta$, then given equation (IA.27), we use that $M_e \to \infty$ to conclude that $\lim_{n\to\infty} spill_{mkt} = 0$.

If $\gamma\theta < \eta$, then we can use the asymptotics from D.7 and the formula for the market-based spillover from (IA.27):

$$\begin{aligned} spill_{mkt}(n; M_e) &\simeq -\frac{\eta}{\gamma}\frac{1}{f_e}\alpha_\infty^{\eta-\theta\gamma} \cdot n^{\frac{\eta-\theta\gamma}{\gamma(1+\theta)-\eta}} \cdot \left[1 - \left(1 - \alpha_\infty^{-\gamma} n^{\frac{-\gamma}{\gamma(1+\theta)-\eta}}\right)^n\right] \\ &\simeq -\frac{\eta}{\gamma}\frac{1}{f_e}\alpha_\infty^{\eta-\theta\gamma} \cdot n^{\frac{\eta-\theta\gamma}{\gamma(1+\theta)-\eta}} \cdot \left[1 - \exp\left(-\alpha_\infty^{-\gamma} n^{\frac{\gamma\theta-\eta}{\gamma(1+\theta)-\eta}}\right)\right] \\ &\simeq -\frac{\eta}{\gamma}\frac{1}{f_e}\alpha_\infty^{\eta-\theta\gamma} \cdot n^{\frac{\eta-\theta\gamma}{\gamma(1+\theta)-\eta}} \cdot \alpha_\infty^{-\gamma} n^{\frac{\gamma\theta-\eta}{\gamma(1+\theta)-\eta}} \to -\frac{\eta}{\gamma}\frac{1}{f_e}\alpha_\infty^{\eta-\gamma(1+\theta)}. \end{aligned} \tag{IA.31}$$

Using the definition of α_∞ from the proof above, we conclude $\lim_{n\to\infty} spill_{mkt}(n; M_e) = -(\gamma - \eta)/\gamma$.

In the knife-edge case with $\gamma\theta = \eta$, we have

$$spill_{mkt}(n; M_e) \simeq -\frac{\eta}{\gamma}\frac{1}{f_e} \cdot \left[1 - \left(1 - \alpha_\infty n^{-1}\right)^n\right] \to -\frac{\eta}{\gamma}\frac{1}{f_e} \cdot e^{-\alpha_\infty}. \tag{IA.32}$$

We can bound the market-based spillover in the limit: $\lim_{n\to\infty} |spill_{mkt}(n; M_e)| < \alpha_\infty^{-1} \cdot (\gamma - \eta)/\gamma$. ∎

Our conclusions are therefore robust to including competition to enter. Intuitively, marginal firms drive the externality in both settings. In our baseline, the externality operates at the extensive margin: more entry displaces the profits of excluded marginal firms. In this model, the externality is on the intensive margin: firm entry increases the productivity of the marginal firm and thus advertisement costs for all producing firms.

D.3 Microfoundation for Profits

In our baseline model, we assumed the profit function of equation (4). We now endogenize how profits $\pi(a)$ are determined in equilibrium, which creates new sources of spillovers First, when a firm innovates, it not only affects shareholders but also workers (see, e.g., Kogan et al. (2020) for recent evidence). Next, a new product alters consumers' choice over their entire consumption basket (Blanchard and Kiyotaki (1987)). Finally, others can learn from this innovation, as emphasized by Romer (1986). These spillovers, while important, do not change qualitatively the aggregate behavior of the economy in response to speculation. Specifically, an increase in disagreement n always yields a bubble and increases the market-based private value of innovation (in absolute terms and relatively to the outcome-based measure).

In what follows, it will be convenient to define:

$$\mathcal{I}_n(M_e;\eta) = \int_{F(1-\frac{M_e}{M})}^{\infty} a^{\eta} dF(a). \tag{IA.33}$$

In the baseline model, $\mathcal{I}_n = V^{(n)}$.The integral with no disagreement is $\mathcal{I}_1$. We will use $\mathcal{I}_n$ when the dependence of the integral to M_e or σ is unambiguous. Under the Pareto distribution with parameter γ, we have the following result:

$$\mathcal{I}_1 = \frac{\gamma}{\gamma-\sigma} \cdot \left(\frac{M_e}{M}\right)^{\frac{\sigma}{\gamma}-1}. \tag{IA.34}$$

D.3.1 Models

Labor with decreasing returns to scale. A simple way to endogenize the profit function is to assume households are endowed with a fixed quantity of labor L, which firms use to produce a homogeneous good according to a decreasing returns to scale production function:

$$y(a) = a \cdot \frac{\sigma}{\sigma-1} \ell^{\frac{\sigma-1}{\sigma}}, \tag{IA.35}$$

where y is firm output, ℓ is firm labor input, and the parameter $\sigma \in [1,\infty]$ controls the returns to scale in labor.

Firms solve

$$\max_{\ell(a)} \pi(a) = a \cdot \frac{\sigma}{\sigma-1} \ell(a)^{\frac{\sigma-1}{\sigma}} - w\ell(a)$$

taking the equilibrium wage w as given. The firm's first-order conditions imply:

$$\pi(a) = \frac{1}{\sigma-1} \cdot w^{1-\sigma} a^{\sigma}. \tag{IA.36}$$

As in Section 3, profits are isoelastic with respect to productivity a.

Given the equilibrium quantities, we can decompose aggregate output into the profit and labor shares. First, observe that aggregate ouput is:

$$\mathcal{C} = M_e \cdot \int_{\underline{a}}^{\infty} y(a) dF(a) = M_e \cdot \frac{\sigma}{\sigma-1} w^{1-\sigma} \mathcal{I}_1.$$

From this expression we can simplify the ex-ante valuation of firms:

$$\begin{aligned} V^{(n)}(M_e) &= \int_{\underline{a}}^{\infty} \pi(a) dF^n(a) = \frac{1}{\sigma - 1} \cdot w^{1-\sigma} \cdot \mathcal{I}_n(M_e) \\ &= \frac{1}{\sigma} \cdot \frac{\mathcal{C}}{M_e} \cdot \frac{\mathcal{I}_n(M_e)}{\mathcal{I}_1(M_e)}. \end{aligned}$$

In addition, market clearing on the input market yields:

$$L = M_e \cdot w^{-\sigma} \int_{\underline{a}}^{\infty} a^{\sigma} dF(a) = M_e \cdot w^{-\sigma} \cdot \mathcal{I}_1, \tag{IA.37}$$

Thus we can decompose consumption for household j into labor income and profits from its investment:

$$\mathcal{C}_j = \underbrace{\frac{\sigma - 1}{\sigma} \cdot \mathcal{C}}_{\text{labor income: } wL} + \underbrace{\frac{1}{\sigma} \cdot \frac{\mathcal{I}_n}{\mathcal{I}_1} \cdot \mathcal{C}}_{\text{firm profits: } V^{(n)}} . \tag{IA.38}$$

The equilibrium condition that determines entry in equilibrium is:

$$W'(M_e) = \frac{1}{\sigma} \cdot \frac{\mathcal{C}}{M_e} \cdot \frac{\mathcal{I}_n(M_e)}{\mathcal{I}_1(M_e)}. \tag{IA.39}$$

Demand externalities. An additional concern is that goods may not be perfect substitutes. To that end, we consider monopolistic competition at date 1 in the style of Dixit and Stiglitz (1977). In particular, each firm produces a differentiated variety indexed by i, and household utility over the set of goods produced is:

$$\mathcal{C} = \left(\int_0^{M_e} \int_{F^{-1}\left(1 - \frac{M}{M_e}\right)}^{\infty} c\,(a,i)^{\frac{\sigma-1}{\sigma}}\, dF(a) di \right)^{\frac{\sigma}{\sigma-1}}. \tag{IA.40}$$

Firms operate a linear technology in labor, and output for a firm with productivity a is $y = a\ell$. We leave our other assumptions unaltered.

At date 1, household j with total expenditure E_j solves:

$$\begin{aligned} \mathcal{C}(E_j) = \max_{\{c(a,i)\}} & \left(\int_0^{M_e} \int_{F^{-1}\left(1 - \frac{M}{M_e}\right)}^{\infty} c\,(a,i)^{\frac{\sigma-1}{\sigma}}\, dF(a) di \right)^{\frac{\sigma}{\sigma-1}} \\ \text{s.t.} & \int_0^{M_e} \int_{F^{-1}\left(1 - \frac{M}{M_e}\right)}^{\infty} p\,(a,i)\, c\,(a,i)\, dF(a) di \le E_j, \end{aligned}$$

which yields the demand curve:

$$c(p) = \mathcal{C} p^{-\sigma}.$$

Firms maximize profits by setting prices, taking as given the demand curve from each household:

$$\max_{p(a)} \; p(a)y\left(p\left(a\right)\right) - \frac{wy\left(p\left(a\right)\right)}{a} = \mathcal{C}\left[p(a)^{1-\sigma} - \frac{w}{a}p(a)^{-\sigma}\right].$$

The optimal price is therefore

$$p(a) = \frac{\sigma}{\sigma-1}\frac{w}{a}.$$

We can then compute output y, revenue py, labor expenditure $w\ell$, and profits π as functions of productivity:

$$\begin{aligned}
y &= \mathcal{C}w^{-\sigma}a^{\sigma}\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} \\
py &= \mathcal{C}w^{1-\sigma}a^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \\
w\ell &= \frac{\sigma-1}{\sigma}\mathcal{C}w^{1-\sigma}a^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \\
\pi &= \frac{1}{\sigma}\mathcal{C}w^{1-\sigma}a^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}.
\end{aligned}$$

We see that labor expenditure is a fraction $(\sigma-1)/\sigma$ of revenues, and profits make up the remaining $1/\sigma$ share.

Labor market clearing gives $\mathcal{C}(\sigma-1)/\sigma = wL$. In equilibrium, aggregate expenditure is equal to aggregate consumption, so we have:

$$\begin{aligned}
\mathcal{C} &= \mathcal{C}\left(\frac{\sigma}{\sigma-1}w\right)^{1-\sigma} M_e\mathcal{I}_1(M_e, \sigma-1) \\
&= M_e^{\frac{1}{\sigma-1}}\left(\frac{\gamma}{\gamma-(\sigma-1)}\right)^{\frac{1}{\sigma-1}}\left(\frac{M_e}{M}\right)^{\frac{(\sigma-1)-\gamma}{(\sigma-1)\gamma}} \cdot L \\
&= \left(\frac{\gamma}{\gamma-(\sigma-1)}\right)^{\frac{1}{\sigma-1}} M^{\frac{1}{\sigma-1}-\frac{1}{\gamma}} M_e^{\frac{1}{\gamma}} \cdot L.
\end{aligned}$$

Therefore, we have $\mathcal{E}_{\mathcal{C}} = \mathcal{E}_w = 1/\gamma$. Alternatively, notice that the labor allocation is efficient with monopolistic competition and the aggregate production function is homogeneous of degree one in the distribution of productivity. Because an increase in M_e increases all productivities with an elasticity $1/\gamma$, this results in an elasticity of aggregate consumption of $1/\gamma$.

Knowledge spillovers. As emphasized by Bloom, Schankerman, and Van Reenen (2013), firms also learn from each other's innovations. We capture the role of knowledge spillovers following Romer (1990), by assuming that a firm's productivity combines its own type, a, and an aggregate of all the active firms' productivity, A:

$$y = \frac{\sigma}{\sigma-1}a^{1-\alpha}A^{\alpha}\ell^{\frac{\sigma-1}{\sigma}}, \tag{IA.41}$$

where α is the intensity of knowledge spillovers, which is zero in (IA.35) . We assume the aggregator is homogeneous of degree one in the productivity distribution.

We use a Hölder mean of the productivity of all firms producing:

$$A = \left(\frac{M_e}{M} \int_{\left(\frac{M_e}{M}\right)^{\frac{1}{\gamma}}}^{\infty} a^q dF(a) \right)^{\frac{1}{q}}.$$

Imposing $q < \gamma$ so that the integral is well-defined, we have:

$$A = \left(\frac{\gamma}{\gamma - q} \right)^{\frac{1}{q}} \left(\frac{M_e}{M} \right)^{\frac{1}{\gamma}}.$$

Our results generalize to any aggregator that is homogeneous of degree one in the productivity distribution of producing firms. Such aggregators similarly yield an elasticity $1/\gamma$ with respect to M_e.

Firms maximize their profits, taking the wage as given:

$$\max_{\ell} \frac{\sigma}{\sigma - 1} a^{1-\alpha} A^{\alpha} \ell^{\frac{\sigma-1}{\sigma}} - w\ell.$$

The demand for labor is therefore

$$\ell = \left(\frac{w}{a^{1-\alpha} A^{\alpha}} \right)^{-\sigma},$$

and we have:

$$\begin{aligned} y(a) &= \frac{\sigma}{\sigma - 1} \left(a^{1-\alpha} A^{\alpha} \right)^{\sigma} w^{1-\sigma} \\ w\ell(a) &= \left(a^{1-\alpha} A^{\alpha} \right)^{\sigma} w^{1-\sigma} = \frac{\sigma - 1}{\sigma} y(a) \\ \pi(a) &= \frac{1}{\sigma - 1} \left(a^{1-\alpha} A^{\alpha} \right)^{\sigma} w^{1-\sigma} = \frac{1}{\sigma} y(a). \end{aligned}$$

The labor share is still $(\sigma-1)/\sigma$, but the microeconomics of the firms' interactions differ, as can be seen from the expression for profits $\pi(a)$.

The market-clearing condition for labor is:

$$\begin{aligned} L &= w^{-\sigma} A^{\alpha\sigma} M_e \mathcal{I}_1 \left(M_e, (1-\alpha)\sigma \right) \\ &= \left(\frac{\gamma}{\gamma - q} \right)^{\frac{\alpha\sigma}{q}} \left(\frac{M_e}{M} \right)^{\frac{\alpha\sigma}{\gamma}} M_e \frac{\gamma}{\gamma - (1-\alpha)\sigma} \left(\frac{M_e}{M} \right)^{\frac{(1-\alpha)\sigma}{\gamma} - 1} w^{-\sigma} \\ &= \left(\frac{\gamma}{\gamma - q} \right)^{\frac{\alpha\sigma}{q}} \frac{\gamma}{\gamma - (1-\alpha)\sigma} M \left(\frac{M_e}{M} \right)^{\frac{\sigma}{\gamma}} w^{-\sigma} \\ w &= \left(\frac{M}{L} \right)^{\frac{1}{\sigma}} \left(\frac{\gamma}{\gamma - q} \right)^{\frac{\alpha}{q}} \left(\frac{\gamma}{\gamma - (1-\alpha)\sigma} \right)^{\frac{1}{\sigma}} \left(\frac{M_e}{M} \right)^{\frac{1}{\gamma}}. \end{aligned}$$

We still have $(\sigma - 1)/\sigma \mathcal{C} = wL$, and the same elasticities: $\mathcal{E}_w = \mathcal{E}_{\mathcal{C}} = \mathcal{E}_A = 1/\gamma$.

D.3.2 Spillovers

General formula. We now derive the total spillovers, incorporating the additional externalities in the models above

Proposition D.1. *For all the models of Section D.3, the market-based spillover is:*

$$spill_{mkt}(n) = \underbrace{\mathcal{E}_{\mathcal{I}_n}}_{bus.\ stealing} + \underbrace{\mathcal{E}_{\pi}}_{general\ eq.} + \underbrace{(\sigma-1)\mathcal{E}_{\mathcal{C}}\,\frac{\mathcal{I}_1}{\mathcal{I}_n}}_{appropriability}. \tag{IA.42}$$

The outcome-based spillover, which does not depend on disagreement, is:

$$spill_{out} = \mathcal{E}_{\mathcal{I}_1} + \mathcal{E}_{\pi} + (\sigma-1)\mathcal{E}_{\mathcal{C}}. \tag{IA.43}$$

This decomposition highlights how disagreement can matter for market-based measures of spillovers. While our theoretical exercise is not exhaustive, most spillovers considered in the innovation literature fit in one of our three categories.

The first category is business-stealing. As discussed previously, disagreement dampens the effect of business-stealing. Competitive interactions between firms of different productivities can take different forms than the displacement of our model. Nevertheless, spillovers that are more bottom-heavy tend to be dissipated by disagreement in general.

This stands in contrast to the second category: general equilibrium effects. In our models, these are the effects of the wage, aggregate demand, and aggregate knowledge on firm profits. Each of these affect all firms proportionally regardless of productivity and can be summarized by the elasticity of firm profits to firm entry, holding productivity constant, $\mathcal{E}_{\pi}$. Because the response to these general equilibrium forces does not interact with the productivity distribution, these spillovers do not depend on beliefs.

Finally, there are appropriability effects accruing to workers. Because the surplus of workers is determined in the spot market for labor, it does not depend on the relative positions of firms. Unlike firm valuations, wage expectations are not affected by speculation about the relative positions of firms beyond its direct impact on entry and overall labor demand. When disagreement increases, the market-based spillover to workers disappears. This insight is not specific to workers. Rather, it affects all stakeholders of the innovation process that our model abstracts from, including owners of other production inputs in scarce supply or consumers who enjoy some of the surplus.

Labor with decreasing returns to scale. While the business-stealing effect remains, there are two additional sources of spillovers in this model. First, there is an *appropriability effect* as investors do not account for workers capturing some of the surplus from innovation through higher wages. Second, there is an *input price effect* as competition of firms for the same source of labor pushes wages up, making labor more expensive for every other firm.

From the decomposition in equation (IA.38), we can write the market-based social value as:

$$\frac{1}{\sigma} \cdot \frac{d\left(\mathcal{I}_n/\mathcal{I}_1 \cdot \mathcal{C}\right)}{dM_e} + \frac{\sigma-1}{\sigma} \cdot \frac{d\mathcal{C}}{dM_e}. \tag{IA.44}$$

As before, the corresponding spillover is a ratio of the direct and indirect effect of firm entry:

$$1 - spill_{mkt}(n) = \frac{M_e}{\mathcal{C}}\frac{\mathcal{I}_1}{\mathcal{I}_n} \cdot \frac{d\left(\mathcal{I}_n/\mathcal{I}_1 \cdot \mathcal{C}\right)}{dM_e} + (\sigma-1) \cdot \frac{M_e}{\mathcal{C}}\frac{\mathcal{I}_1}{\mathcal{I}_n} \cdot \frac{d\mathcal{C}}{dM_e},$$

which implies:

$$spill_{mkt}(n) = -\mathcal{E}_{\mathcal{I}_n} + (\sigma-1)\,\mathcal{E}_{\mathcal{C}} - (\sigma-1)\mathcal{E}_{C} \cdot \frac{\mathcal{I}_1}{\mathcal{I}_n}. \tag{IA.45}$$

When focusing on the outcome-based measure, we can just replace n with 1.

The first term captures the business-stealing effect of the model in Section 3, which is still present in this economy with the same magnitude $\mathcal{E}_{\mathcal{I}_n}$. Hence, the predictions for this spillover from Section 4.2 still hold. However, the presence of workers introduces two new sources of spillovers.

The second term captures the input price effect. The functional form arises from two facts. First, from equation (IA.36), the elasticity of profits with respect to the wage is $1-\sigma$. Next, $\mathcal{E}_{\mathcal{C}} = \mathcal{E}_w$ since the constant labor share and labor supply implies that the wage grows as fast as aggregate output. Disagreement does not impact the input price spillover because change in wage affects all firms proportionally, irrespective of productivity. This negative externality is larger when firms rely more on labor—high σ—or when the economy responds more to entry—high $\mathcal{E}_{\mathcal{C}}$.

The third term captures the appropriability effect. When new firms enter, aggregate output increases, with constant elasticity $\mathcal{E}_{\mathcal{C}} = 1/\gamma$. The production function implies that workers receive a constant fraction of aggregate output given by the labor share $(\sigma-1)/\sigma$. Therefore, the social value of entry for workers is $\frac{\sigma-1}{\sigma}\mathcal{E}_{\mathcal{C}}\mathcal{C}/M_e$. Importantly, this quantity does not depend directly on the amount of disagreement: all firms offer the same wage in equilibrium, so beliefs are irrelevant for workers. However, the market-based private value of the firm does. This value is $\mathcal{I}_n/\mathcal{I}_1 \times \sigma^{-1}\mathcal{C}/M$: the relative expected output of a favorite firm to an average firm multiplied by average profits of firms. The appropriability effect is a positive spillover, larger when workers capture more of total surplus—high σ—or when entry has a stronger impact on output—high $\mathcal{E}_{\mathcal{C}}$. Disagreement does not affect the outcome-based spillover. In contrast, the market-based spillover decreases in n, disappearing altogether in the limit when $n \to \infty$. This is because the bubble inflates the market value of firms, but workers' acknowledge that their earnings are determined by the average producing firm and realize that not all firms will be winners.[5]

The asymptotics for the business-stealing effect are similar to that of the

[5] In the case of agreement, $n = 1$, the appropriability and input price spillovers exactly cancel out. It is a situation where pecuniary externalities cancel out even though the first welfare theorem does not hold because of business-stealing.

model of Section 3.

Lemma D.9 (Asymptotics for business-stealing effect)**.** *In the high-disagreement limit ($n \to \infty$), the business-stealing effect converges to a limit that depends on the marginal cost of firm creation θ:*

- *If $\theta\gamma < 1$, then $\lim_{n\to\infty} \mathcal{E}_{\mathcal{I}_n} = \mathcal{E}_{\mathcal{I}_1} = \frac{\sigma}{\gamma} - 1$.*
- *If $\theta\gamma > 1$, then $\lim_{n\to\infty} \mathcal{E}_{\mathcal{I}_n} = 0$.*
- *If $\theta\gamma = 1$, then $\lim_{n\to\infty} \mathcal{E}_{\mathcal{I}_n} = \alpha_\infty e^{-\alpha_\infty}/f_e$.*

Proof. The free-entry condition equation (IA.39) leads to:

$$(\sigma - 1)\left(\frac{\gamma - \sigma}{\gamma} L\right)^{\frac{1-\sigma}{\sigma}} \cdot f_e = M^{\theta + \frac{\sigma-\gamma}{\gamma}\frac{\sigma-1}{\sigma}} \cdot M_e^{\frac{1-\sigma}{\gamma} - \theta} \cdot \mathcal{I}_n.$$

We recast the free-entry condition using $\underline{a}$ to be able to use the asymptotic results from Lemma C.2

$$\text{constant} = \underline{a}^{1-\sigma-\gamma\theta} \int_{\underline{a}}^{\infty} x^\sigma dF_n(x).$$

Writing $\tilde{\theta} = \theta + (\sigma - 1)/\gamma$ and $\tilde{\eta} = \sigma$, we recognize the expression from Lemma C.2 and use Proposition C.1. ∎

For the labor surplus term, we study the behavior of $\mathcal{I}_1/\mathcal{I}_n$.

Lemma D.10 (Asymptotics for labor surplus distortion)**.** *In the high-disagreement limit ($n \to \infty$), the labor surplus distortion disappears:*

$$\lim_{n\to\infty} (\sigma - 1)\mathcal{E}_{\mathcal{C}} \frac{\mathcal{I}_1}{\mathcal{I}_n} = 0$$

Proof. Since $\tilde{\theta} > 0$, Lemma C.1 gives $\lim_{n\to\infty} M_e = \infty$. The proof of Lemma C.1 implies $\lim_{n\to\infty} \mathcal{I}_n = \infty$. Finally, because $\sigma < \gamma$,

$$\mathcal{I}_1 = \frac{\gamma}{\gamma - \sigma}\left(\frac{M_e}{M}\right)^{\frac{\sigma-\gamma}{\gamma}} \to 0$$

as $n \to \infty$. Therefore, $\mathcal{I}_1/\mathcal{I}_n$ converges to 0. ∎

Demand externalities. At the aggregate level, the economy behaves similarly to the previous model.[6] However, the microeconomics of firms' interactions is different and so are profits:

$$\pi(a) = \frac{1}{\sigma} \cdot \left(\frac{\sigma}{\sigma - 1}\right)^{1-\sigma} w^{1-\sigma} \cdot \mathcal{C} \cdot a^{\sigma-1}. \tag{IA.46}$$

[6]The profit share is $1/\sigma$, the aggregate production function is homogeneous of degree one in the distribution of productivities, and the relative labor allocations are efficient. This result was first shown in Lerner (1934). It is the consequence of the homogeneous distortions at the firm level when markups are constant. The macroeconomic elasticities of aggregate consumption and wages to firm entry are therefore $\mathcal{E}_{\mathcal{C}} = \mathcal{E}_w = 1/\gamma$.

The elasticity of profits to individual firm productivity is $\sigma - 1$ instead of σ. In addition, profits are now increasing in aggregate demand $\mathcal{C}$ because of imperfect substitution across goods.

This role of aggregate demand gives rise to an additional source of spillovers: a demand externality as in Blanchard and Kiyotaki (1987). Similar to the role of the wage, aggregate demand affects all firms proportionally. Therefore the demand spillover is the product of the elasticity of aggregate output to entry $\mathcal{C}$ and the elasticity of profits to aggregate output:

$$\text{Demand Spillover} = \mathcal{E}_{\mathcal{C}}. \tag{IA.47}$$

Demand spillovers do not depend on disagreement and are identical whether measured using outcomes or market value.

Knowledge spillovers. Like the previous two cases, the impact of knowledge on profits is the same irrespective of each individual firm's productivity. The knowledge spillover is therefore the product of the elasticity of profit to knowledge $\alpha\sigma$, and the elasticity of knowledge to entry $1/\gamma$[7]

$$\text{Knowledge Spillover} = \alpha(\sigma - 1)\mathcal{E}_{\mathcal{C}}. \tag{IA.48}$$

This expression does not depend on disagreement and is identical for market-based and outcome-based measures.

D.3.3 Three Illustrations of the Role of Disagreement

We draw three implications from Proposition D.1 that illustrate that disagreement fundamentally alters how to measure and interpret the value of innovation.

Macroeconomic versus Microeconomic Elasticities. Without disagreement, market-based and outcome-based measures of spillovers coincide because valuations are expected outcomes. In the economies we have considered, the result is even stronger. The spillover under agreement is the same across all specifications: with labor only, with aggregate demand, and with aggregate knowledge:

$$spill_{out} = spill_{mkt}(1) = \sigma\mathcal{E}_{\mathcal{C}} - 1. \tag{IA.49}$$

While we can derive this expression from equation (IA.43) separately for each model, a simple macroeconomic argument justifies the result. The total effect of entry is the response of aggregate output to entry $d\mathcal{C}/dM_e$. Ex ante, each firm contributes an equal fraction to total output, and the value of a firm is output times the profit share $1/\sigma$. Hence, the value of a firm is $\mathcal{C}/(M_e\sigma)$, and the spillover is given immediately by equation (IA.49).

Regardless of the nature of firm interactions, only two macroeconomic quantities are needed to evaluate the total spillover—the capital share and the elasticity

[7]Because both knowledge and output are determined by the distribution of firm productivity, they grow at the same pace with entry $\mathcal{E}_A = \mathcal{E}_{\mathcal{C}}$.

of aggregate output to firm entry. In particular, all our specifications lead to the same values of these two quantities. Disagreement breaks this result. Because different spillovers respond differently to disagreement, the aggregate reasoning under agreement no longer works.

One particularly telling example is the limit of large n when the market-based spillover converges to the profit elasticity $\mathcal{E}_\pi$. This spillover measure is a microeconomic elasticity. It is the response of the profits of one specific firm (i.e., of given productivity) to overall entry. This implies that one needs firm-level data rather than aggregate data to estimate spillovers. Moreover, across our three model specifications, while the outcome-based spillover is identical, the market-based spillover for large disagreement takes different values:

$$spill_{mkt}(n \to \infty) = -\frac{\sigma - 1}{\gamma} \qquad \text{with labor only,} \tag{IA.50}$$

$$spill_{mkt}(n \to \infty) = -\frac{\sigma - 2}{\gamma} \qquad \text{with aggregate demand,} \tag{IA.51}$$

$$spill_{mkt}(n \to \infty) = -\frac{(1-\alpha)\sigma - 1}{\gamma} \qquad \text{with aggregate knowledge.} \tag{IA.52}$$

In other words, the nature of microeconomic interactions matters for market-based spillovers in the presence of disagreement.

Reversal of Comparative Statics. Key properties of the economy often have an opposite impact on the total spillover, depending on whether it is measured using outcomes or market values. The following proposition highlights one such reversal, for a parameter common to all of our specification, σ.

Proposition D.2. *For all models, the outcome-based spillover is increasing in the labor share. Conversely, with high disagreement ($n \to \infty$ and $\theta > 1/\gamma$), the market-based spillover is decreasing in the labor share.*

The outcome-based spillover is given by equation (IA.49). An economy with a larger labor share mechanically has a lower capital share. Thus, the importance of social value relative to the value of one firm is larger. For the market-based spillover, the focus is on the elasticity of individual firm profits to entry. A higher labor share implies higher reliance on labor and therefore stronger negative spillovers through the wage effect. In the model with labor, comparative statics with respect to the thickness of the tail of the productivity distribution γ are also reversed. Figure IA.2 illustrates these results.

Reversal of Sign of the Spillover. We also identify situations where the sign of the total spillover is reversed, where firm entry brings positive externalities according to market-based measures but negative externalities according to outcome-based measures or vice-versa.

Proposition D.3. *With demand externalities or knowledge spillovers, if the labor share is close to zero, the outcome-based spillover is positive and the market-based spillover is negative with large disagreement. The converse happens when the labor share is close to its upper bound.*

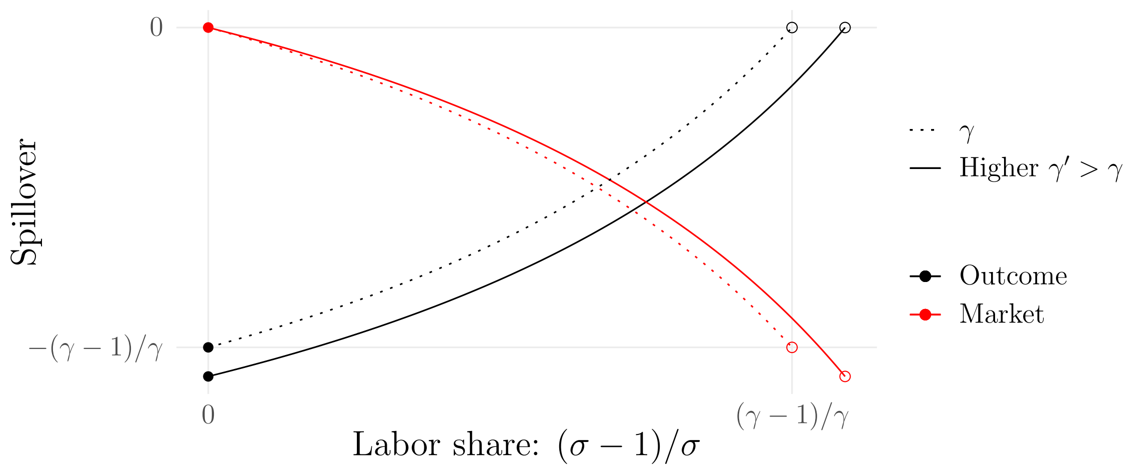

Figure IA.2
Market-based and outcome-based spillovers.
The figure reports the outcome-based (black lines) and market-based spillovers (red lines) as a function of the labor share for the model with labor only. Solid lines correspond to a larger value of γ than dotted lines.

Proof. In both models, the aggregate consumption elasticity is unchanged, the output-based spillover (and market-based spillover under agreement) is unchanged: $spill_{out} = -(\gamma - \sigma)/\gamma$.

In the demand externalities model with speculation, the free-entry condition is:

$$\left(\frac{M_e}{M}\right)^{\theta} = \frac{1}{\sigma}\mathcal{C}\left(\frac{\mathcal{C}}{L}\right)^{1-\sigma}\mathcal{I}_n,$$

which we can rewrite as:

$$KM_e^{\theta-(1-(\sigma-1))/\gamma} = \mathcal{I}_n,$$

where K does not depend on M_e and n. This is again the same condition as the homogeneous goods model, with σ replaced by $\sigma - 1$. The condition for the convergence of $\mathcal{E}_{\mathcal{I}_n}$ from Lemma D.9 still applies as well. In the high-disagreement limit with $\theta > 1/\gamma$, the market-based spillover becomes:

$$spill_{mkt}(n \to \infty) = -1 - \mathcal{E}_{\mathcal{I}_1} + \mathcal{E}_{\mathcal{C}} = -\frac{\sigma - 2}{\gamma}. \tag{IA.53}$$

In the knowledge spillover model, Proposition D.1 and Lemma D.9 still apply, with $\mathcal{I}_1$ and $\mathcal{I}_n$ evaluated with parameter $(1-\alpha)\sigma$. The market-based spillover in the high-disagreement limit with $\theta > 1/\gamma$ becomes:

$$spill_{mkt}(n \to \infty) = -1 - \mathcal{E}_{\mathcal{I}_1} + \mathcal{E}_{\mathcal{C}} = -\frac{(1-\alpha)\,\sigma - 1}{\gamma}. \tag{IA.54}$$

∎

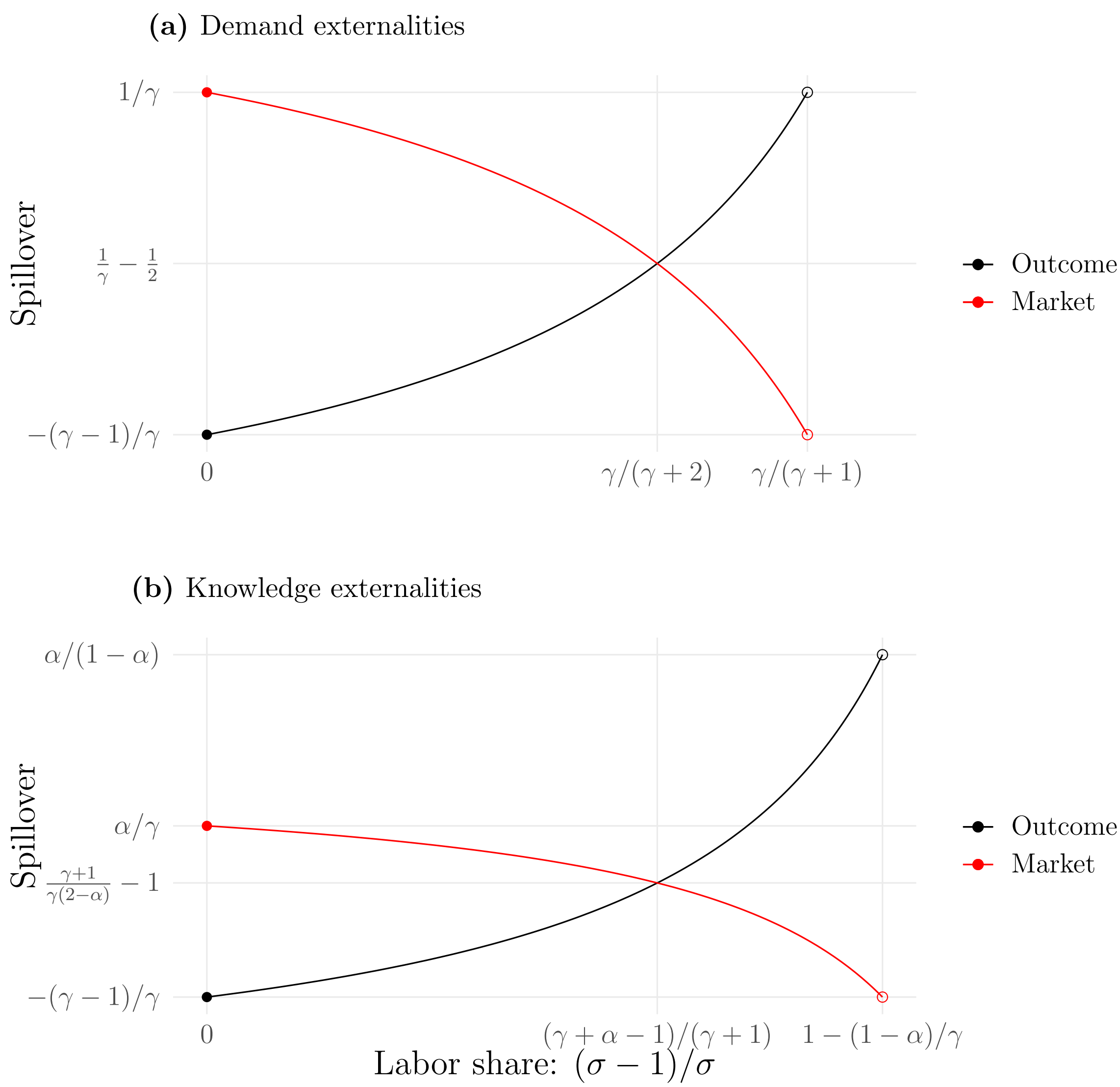

Figure IA.3
Market-based and outcome-based spillovers.

The upper and lower panels of Figure IA.3 shows the sign reversal of the market-based spillover for the model with demand externalities and knowledge spillovers, respectively. When the labor share is low, the labor surplus is relatively small, and the dominant force for the wedge is that firms do not internalize the aggregate decreasing returns to scale of the economy, leading to negative real spillovers. With disagreement however, since firms do not rely much on labor, the general equilibrium effect is small. Hence, the demand or knowledge externality dominates, leading to positive value spillovers. The sign reversal across measures of spillovers is not a knife-edge case. Reversals happen throughout the entire range of the labor share whenever $\gamma = 2$ with demand externalities or $\alpha = 1 - 1/\gamma$ with knowledge spillovers. The proposition also shows that the sign reversal can happen in both directions: a positive spillover becoming negative or a negative spillover becoming positive.

D.4 Variable Number of Participating Firms

D.4.1 Setting and Equilibrium

We study a model where the number of participating firms, M, responds to firm creation M_e, which can be interpreted as households' consumption bundles becoming more or less concentrated as more firms enter the economy. We assume that M varies exogenously with the level of firm entry M_e:

$$M = \frac{1}{M_0^{\chi-1}} \cdot M_e^{\chi},$$

where χ is the elasticity of firms producing to firms created and M_0 a normalization constant. We assume that $\chi \leq 1$ such that we always have $M \leq M_e$.

The cost of creating a firm only depends on M_0 through:

$$W'(M_e) = f_e \left(\frac{M_e}{M_0}\right)^{\theta}.$$

The productivity threshold to produce is now:

$$\underline{a} := F^{-1}\left(1 - \frac{M}{M_e}\right) = \left(\frac{M_e}{M_0}\right)^{\frac{1-\chi}{\gamma}}.$$

The model still features a constant labor share and firm profits are still isoelastic in the productivity:

$$\pi(a) = \frac{1}{\sigma - 1} \cdot w^{1-\sigma} \cdot a^{\sigma} = \frac{1}{\sigma}\frac{\mathcal{C}}{M_e} \cdot \frac{a^{\sigma}}{\mathcal{I}_1},$$

where we have redefined the integrals $\mathcal{I}_1$ and $\mathcal{I}_n$ to adjust for the new expressions for the productivity threshold $\underline{a}$:

$$\mathcal{I}_n(\chi) = \int_{\left(\frac{M_e}{M_0}\right)^{\frac{1-\chi}{\gamma}}}^{\infty} a^{\sigma} dF_n(a)$$

$$\mathcal{I}_1(\chi) = \frac{\gamma}{\gamma - \sigma} \cdot \left(\frac{M_e}{M_0}\right)^{(\chi-1)\frac{\gamma-\sigma}{\gamma}}.$$

The market-clearing condition $L = M_e w^{-\sigma} \mathcal{I}_1$ implies the equilibrium wage:

$$w = \left(\frac{\gamma}{\gamma - \sigma}\right)^{\frac{1}{\sigma}} \cdot L^{-\frac{1}{\sigma}} M_0^{(1-\chi)\left(\frac{1}{\sigma} - \frac{1}{\gamma}\right)} \cdot M_e^{\frac{\chi}{\sigma} + \frac{1-\chi}{\gamma}},$$

so that the labor elasticity is :

$$\mathcal{E}_w = \frac{1}{\gamma} + \chi \cdot \left(\frac{1}{\sigma} - \frac{1}{\gamma}\right).$$

We obtain aggregate consumption by aggregating individual output $\mathcal{C} = M_e\sigma/(\sigma-1)w^{1-\sigma}\mathcal{I}_1$, which yields equilibrium aggregate consumption and elasticity:

$$\begin{aligned}\mathcal{C} &= \frac{\sigma}{\sigma-1}\frac{\gamma}{\gamma-\sigma}\cdot L^{\frac{\sigma-1}{\sigma}}\cdot M_0^{(1-\chi)\left(\frac{1-\sigma}{\sigma}+\frac{\gamma-1}{\gamma}\right)}\cdot M_e^{\frac{1}{\gamma}+\chi\cdot\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)}\\ \mathcal{E}_{\mathcal{C}} &= \mathcal{E}_w = \frac{1}{\gamma}+\chi\cdot\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)\end{aligned}$$

D.4.2 Spillovers

Given the constant labor share and isoelastic profits, we can apply Proposition D.1 and obtain the market-based spillover:

$$spill_{mkt}(n) = \mathcal{E}_{\mathcal{I}_n(\chi)} - 1 - \mathcal{E}_{\mathcal{I}_1(\chi)} - \mathcal{E}_{\mathcal{C}} + (\sigma-1)\mathcal{E}_C\cdot\frac{\mathcal{I}_1(\chi)}{\mathcal{I}_n(\chi)}. \tag{IA.55}$$

From the expression for $\mathcal{I}_n$ we have the following change in the elasticities:

$$\mathcal{E}_{\mathcal{I}_n(\chi)} = (1-\chi)\mathcal{E}_{\mathcal{I}_n(\chi=0)} = (1-\chi)\mathcal{E}_{\mathcal{I}_n} \tag{IA.56}$$

Asymptotics. We now turn to the high-disagreement limit. The first-order condition for firm creation is:

$$\begin{aligned} & f_e\left(\frac{M_e}{M_0}\right)^{\theta} = \frac{1}{\sigma-1}w^{1-\sigma}\mathcal{I}_n(\chi)\\ \iff\quad & \text{constant} = \underline{a}^{-\theta\frac{\gamma}{1-\chi}+(1-\sigma)\frac{\gamma}{1-\chi}\left(\frac{1}{\gamma}+\chi\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)\right)}\int_{\underline{a}}^{\infty}a^{\sigma}dF_n(a).\end{aligned}$$

We define:

$$\tilde{\theta} = \frac{1}{1-\chi}\left(\theta+\frac{\sigma-1}{\gamma}+\chi(\sigma-1)\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)\right),$$

and recognize the entry condition of the baseline model. We apply our previous results, changing the condition for $\mathcal{E}_{\mathcal{I}_n(\chi)}\to 0$ to $\gamma\tilde{\theta}>\sigma$, which reduces to:

$$\gamma(\theta+\chi) > 1+\chi\left(\frac{\gamma}{\sigma}-1\right). \tag{IA.57}$$

If this condition is satisfied, then $\lim_{n\to\infty}\mathcal{E}_{\mathcal{I}_n(\chi)} = 0$. When the inequality is reversed, $\lim_{n\to\infty}\mathcal{E}_{\mathcal{I}_n(\chi)} = \mathcal{E}_{\mathcal{I}_1(\chi)} = (1-\chi)(\sigma-\gamma)/\gamma$. With equality, the elasticity admits a finite limit between these two values.

Behavior of the spillovers. The outcome-based spillover is

$$spill_{out} = -1+\sigma\mathcal{E}_{\mathcal{C}} = -\frac{\gamma-\sigma}{\gamma}+\chi\sigma\left(\frac{1}{\sigma}+\frac{1}{\gamma}\right). \tag{IA.58}$$

The market-based spillover with high disagreement, when θ is large enough, is:

$$spill_{mkt}(n \to \infty) = -1 - \mathcal{E}_{\mathcal{I}_1(\chi)} + \mathcal{E}_{\mathcal{C}} = -\frac{\sigma-1}{\gamma} - \chi(\sigma-1)\left(\frac{1}{\sigma} - \frac{1}{\gamma}\right) \quad \text{(IA.59)}$$

As long as $|\chi| < 1$, we obtain similar results as without a variable number of participating firms. A higher elasticity of firm participation with respect to firm entry leads to opposite results with and without disagreement. A large elasticity χ dampens the outcome-based spillovers because it diminishes business stealing. However it strengthens the market-based spillover with disagreement: in response to firm entry, labor demand responds at the intensive margin with more productive firms and at the extensive margin with more participating firms.

D.5 Participation Costs

Another way to ensure the marginal firm makes zero profit is to assume firms invest in infrastructure to produce. In particular, suppose that upon entry all firms can participate on the goods market, but firms must buy one unit of infrastructure to reach all of their customers. Households produce infrastructure competitively at a cost of effort Φ. In an equilibrium with M producing firms, the price of infrastructure is:

$$\Phi'(M) = \varphi(M) = \varphi_0 \cdot M^{\nu}$$

with $\nu > 0$, so that the cost of infrastructure is increasing in the mass of producing firms M.

D.5.1 Participation Costs in the Baseline Model

Model. Given M_e and M, profits before the infrastructure costs are unchanged from the standard model in Section D.3 with decreasing returns to scale:

$$\pi(a) = \frac{1}{\sigma-1} \cdot w^{1-\sigma} \cdot a^{\sigma}.$$

The equilibrium wage is also unchanged:

$$w = \left(\frac{\gamma}{\gamma-\sigma}\right)^{\frac{1}{\sigma}} \cdot \left(\frac{M_e}{L}\right)^{\frac{1}{\sigma}} \cdot \left(\frac{M_e}{M}\right)^{\frac{1}{\gamma}-\frac{1}{\sigma}}.$$

The marginal firm has productivity $\underline{a}$ and spends all of its profit on infrastructure. Therefore, we have the zero-cutoff-profit condition $\Phi'(M) = \pi(\underline{a})$, which implies:

$$M^{\nu+\frac{1}{\gamma}+\frac{\sigma-1}{\sigma}} = \frac{1}{\varphi_0}\frac{1}{\sigma-1}\left(\frac{\gamma-\sigma}{\gamma}\right)^{\frac{\sigma-1}{\sigma}} L^{\frac{\sigma-1}{\sigma}} \cdot M_e^{\frac{1}{\gamma}},$$

where we use the fact that $\underline{a} = (M_e/M)^{1/\gamma}$. In Section D.4, we specified an exogenous set of producing firms $M = M_e^{\chi}/M_0^{\chi-1}$. This arises endogenously through our cost of infrastructure with

$$\chi = \frac{1}{\gamma}\left(\nu + \frac{1}{\gamma} + \frac{\sigma-1}{\sigma}\right)^{-1}$$

$$M_0^{1-\chi} = \left(\frac{1}{\varphi_0}\frac{1}{\sigma-1}\left(\frac{\gamma-\sigma}{\gamma}\right)^{\frac{\sigma-1}{\sigma}} L^{\frac{\sigma-1}{\sigma}}\right)^{\left(\nu+\frac{1}{\gamma}+\frac{\sigma-1}{\sigma}\right)^{-1}},$$

where the exponent satisfies $\chi \leq 1$.

We can also compute the elasticity $\mathcal{E}_{\mathcal{C}}$:

$$\mathcal{E}_{\mathcal{C}} = \frac{1}{\gamma} + \chi\left(\frac{1}{\sigma} - \frac{1}{\gamma}\right) = \frac{1}{\gamma}\cdot\frac{1+\nu}{1+\nu+1/\gamma-1/\sigma} = \chi\cdot(1+\nu).$$

The equilibrium condition in the competitive equilibrium is:

$$W'(M_e) = \frac{1}{\sigma}\cdot\frac{\mathcal{C}}{M_e}\cdot\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1},$$

where we define the modified $\widetilde{\mathcal{I}}_n$ integral to account for the infrastructure expenditures of the firm:

$$\widetilde{\mathcal{I}}_n(M_e,\chi) = \int_{\left(\frac{M_e}{M_0}\right)^{\frac{1-\chi}{\gamma}}}^{\infty}\left(a^\sigma - \left(\frac{M_e}{M_0}\right)^{\sigma\frac{1-\chi}{\gamma}}\right) dF_n(a).$$

With $n=1$, we have:

$$\widetilde{\mathcal{I}}_1(M_e,\chi) = \frac{\sigma}{\gamma-\sigma}\cdot\left(\frac{M_0}{M_e}\right)^{(1-\chi)\frac{\gamma-\sigma}{\gamma}} = \frac{\sigma}{\gamma}\cdot\mathcal{I}_1(M_e,\chi).$$

Aggregate profits therefore represent a fraction σ/γ of aggregate revenue after labor costs, while aggregate infrastructure costs account for the other $(\gamma-\sigma)/\gamma$. Therefore, aggregate profits represent a share $1/\gamma$ of consumption and aggregate infrastructure costs $1/\sigma - 1/\gamma$.

Spillovers. The market-based social value is now characterized by:

$$\frac{d}{dM_e}\left[\frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1} + \frac{\sigma-1}{\sigma}\mathcal{C} + \underbrace{\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)\mathcal{C}}_{\text{consumption from infrastructure}} - \underbrace{\Phi(M)}_{\text{cost of infrastructure}}\right]$$

$$= \frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}\frac{\widetilde{\mathcal{I}}_n'}{\widetilde{\mathcal{I}}_n} + \frac{1}{\sigma}\mathcal{C}'\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1} - \frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}\frac{\mathcal{I}_1'}{\mathcal{I}_1} + \frac{\sigma-1}{\sigma}\mathcal{C}' + \left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)\mathcal{C}' - \underbrace{\Phi'(M)M}_{\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)\mathcal{C}}\cdot\frac{1}{M}\frac{dM}{dM_e}.$$

The market-based spillover is therefore:

$$spill_{mkt}(n) = \mathcal{E}_{\widetilde{\mathcal{I}}_n} - \mathcal{E}_{\mathcal{I}_1} - 1 + \mathcal{E}_{\mathcal{C}} + (\sigma-1)\mathcal{E}_{\mathcal{C}} \cdot \frac{\mathcal{I}_1}{\widetilde{\mathcal{I}}_n} + \underbrace{\frac{\gamma-\sigma}{\gamma}(\mathcal{E}_{\mathcal{C}} - \mathcal{E}_M) \cdot \frac{\mathcal{I}_1}{\widetilde{\mathcal{I}}_n}}_{\substack{\text{surplus from}\\ \text{infrastructure costs}}},$$

where the last term accounts for the surplus from infrastructure creation.

The outcome-based spillover (and market-based spillover under agreement) is:

$$spill_{out} = -1 + \mathcal{E}_{\mathcal{C}} + \left[(\sigma-1)\mathcal{E}_{\mathcal{C}} + \left(1 - \frac{\sigma}{\gamma}\right)(\mathcal{E}_{\mathcal{C}} - \chi)\right]\frac{\gamma}{\sigma}.$$

Using the values of $\mathcal{E}_{\mathcal{C}}$ and χ, we obtain:

$$spill_{out} = -1 + \chi\gamma\left(1 + \nu - \frac{1}{\sigma} + \frac{1}{\gamma}\right) = 0, \tag{IA.60}$$

given the formula above for χ. Participation is now a good traded on a competitive market. Hence the first welfare theorem applies, and there are no outcome-based spillovers.

Now we apply a similar reasoning as earlier to find the condition for convergence when θ is large. The condition for convergence of $\mathcal{E}_{\widetilde{\mathcal{I}}_n}$ is the same as for $\mathcal{E}_{\mathcal{I}_n}$:

$$\gamma(\theta + \chi) > 1 + \chi\left(\frac{\gamma}{\sigma} - 1\right) \tag{IA.61}$$

$$\iff \quad \gamma\theta > 1 + \frac{1}{\nu + \frac{1}{\gamma} + \frac{\sigma-1}{\sigma}}\left(\frac{1}{\sigma} - \frac{1}{\gamma} - 1\right). \tag{IA.62}$$

As $n \to \infty$, we have that $\widetilde{\mathcal{I}}_n \to \infty$ and $\mathcal{I}_1 \to 0$, and therefore, $\mathcal{I}_1/\widetilde{\mathcal{I}}_n \to 0$.

For the high-disagreement market-based spillover we have

$$spill_{mkt}(n \to \infty) = -(\sigma-1) \cdot \mathcal{E}_w = -\frac{\sigma-1}{\gamma} - (\sigma-1)\chi\left(\frac{1}{\sigma} - \frac{1}{\gamma}\right) \tag{IA.63}$$

$$= -\frac{\sigma-1}{\gamma} \cdot \left(\frac{1+\nu}{1+\nu+1/\gamma-1/\sigma}\right). \tag{IA.64}$$

We find that $|spill_{mkt}(n \to \infty)|$ is decreasing in ν since $1/\gamma < 1/\sigma$. As the cost of producing infrastructure becomes steeper, the sensitivity of firm participation to firm creation is smaller, and the market-based spillover is less responsive. In the limit with $\nu \to \infty$, a fixed number of firms produces, and we are back to our baseline $spill_{mkt}(n \to \infty) = -(\sigma-1)/\gamma$. Finally, $|spill_{mkt}(n \to \infty)|$ is increasing in σ.

D.5.2 Melitz (2003) Model: Participation Costs and Dixit-Stiglitz

We now introduce Dixit-Stiglitz preferences to the above model, as in Melitz (2003).

Model. Given M_e and M, profits before the infrastructure costs are unchanged from the standard model with decreasing returns to scale:

$$\pi(a) = \frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \cdot \mathcal{C} \cdot w^{1-\sigma} \cdot a^{\sigma-1}.$$

The equilibrium consumption is also unchanged:

$$\frac{\mathcal{C}}{L} = \left(\frac{\gamma}{\gamma-(\sigma-1)}\right)^{\frac{1}{\sigma-1}} M^{\frac{1}{\sigma-1}-\frac{1}{\gamma}} M_e^{\frac{1}{\gamma}}.$$

The marginal firm has productivity $\underline{a}$ and spends all of its profit on infrastructure. Therefore, we have the zero-cutoff-profit condition $\Phi'(M) = \pi(\underline{a})$, which implies

$$M^{\nu+\frac{1}{\gamma}+\frac{\sigma-2}{\sigma-1}} = \frac{1}{\varphi_0}\frac{1}{\sigma}\left(\frac{\gamma-(\sigma-1)}{\gamma}\right)^{\frac{\sigma-2}{\sigma-1}} L \cdot M_e^{\frac{1}{\gamma}},$$

where we use the fact that $\underline{a} = (M_e/M)^{1/\gamma}$. In Section D.4, we specified an exogenous set of producing firms $M = M_e^{\chi}/M_0^{\chi-1}$. This arises endogenously through our cost of infrastructure with

$$\chi = \frac{1}{\gamma}\left(\nu+\frac{1}{\gamma}+\frac{\sigma-2}{\sigma-1}\right)^{-1},$$

$$M_0^{1-\chi} = \left(\frac{1}{\varphi_0}\frac{1}{\sigma}\left(\frac{\gamma-(\sigma-1)}{\gamma}\right)^{\frac{\sigma-2}{\sigma-1}} L\right)^{\left(\nu+\frac{1}{\gamma}+\frac{\sigma-2}{\sigma-1}\right)^{-1}},$$

where the exponent satisfies $\chi \leq 1$ if and only if $\nu + \frac{\sigma-2}{\sigma-1} \in (-\infty, -1/\gamma) \cup [0, \infty)$. Otherwise, all firms participate as M_e grows to infinity.

Finally, we derive the elasticity $\mathcal{E}_{\mathcal{C}}$:

$$\mathcal{E}_{\mathcal{C}} = \frac{1}{\gamma} + \chi\left(\frac{1}{\sigma-1} - \frac{1}{\gamma}\right) = \frac{1}{\gamma} \cdot \frac{1+\nu}{1+\nu+1/\gamma-1/(\sigma-1)} = \chi \cdot (1+\nu).$$

The equilibrium condition in the competitive equilibrium is:

$$W'(M_e) = \frac{1}{\sigma} \cdot \frac{\mathcal{C}}{M_e} \cdot \frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}.$$

Aggregate profits represent a fraction $(\sigma-1)/\gamma$ of aggregate revenue after labor costs, and aggregate infrastructure costs account for the other $(\gamma-(\sigma-1))/\gamma$. Therefore, aggregate profits represent a share $(\sigma-1)/(\sigma\gamma)$ of consumption and aggregate infrastructure costs $(\gamma-(\sigma-1))/(\sigma\gamma)$.

Spillovers. The market-based social value is:

$$\begin{aligned}
&\frac{d}{dM_e}\left[\frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1} + \frac{\sigma-1}{\sigma}\mathcal{C} + \left(\frac{\gamma-(\sigma-1)}{\sigma\gamma}\right)\mathcal{C} - \Phi(M)\right] \\
=\ &\frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}\frac{\widetilde{\mathcal{I}}'_n}{\widetilde{\mathcal{I}}_n} + \frac{1}{\sigma}\mathcal{C}'\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1} - \frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}\frac{\mathcal{I}'_1}{\mathcal{I}_1} + \frac{\sigma-1}{\sigma}\mathcal{C}' + \left(\frac{\gamma-(\sigma-1)}{\sigma\gamma}\right)\mathcal{C}' - \Phi'(M)M\frac{1}{M}\frac{dM}{dM_e}.
\end{aligned}$$

The market-based spillover is therefore:

$$spill_{mkt}(n) = \mathcal{E}_{\widetilde{\mathcal{I}}_n} - \mathcal{E}_{\mathcal{I}_1} - 1 + \mathcal{E}_{\mathcal{C}} + \left[(\sigma-1)\mathcal{E}_{\mathcal{C}} + \left(1-\frac{\sigma-1}{\gamma}\right)(\mathcal{E}_{\mathcal{C}}-\chi)\right]\frac{\mathcal{I}_1}{\widetilde{\mathcal{I}}_n}. \tag{IA.65}$$

The outcome-based spillover (and market-based spillover under agreement) is:

$$\begin{aligned}
spill_{out} &= -1 + \mathcal{E}_{\mathcal{C}} + \left[(\sigma-1)\mathcal{E}_{\mathcal{C}} + \left(1-\frac{\sigma-1}{\gamma}\right)(\mathcal{E}_{\mathcal{C}}-\chi)\right]\frac{\gamma}{\sigma-1} && \text{(IA.66)} \\
&= \frac{1}{\sigma-1}\cdot\frac{1+\nu}{1+\nu+1/\gamma-1/(\sigma-1)}. && \text{(IA.67)}
\end{aligned}$$

We apply the reasoning again to find the condition for convergence when θ is large. The condition for convergence of $\mathcal{E}_{\widetilde{\mathcal{I}}_n}$ is the same as for $\mathcal{E}_{\mathcal{I}_n}$:

$$\begin{aligned}
&\gamma(\theta+\chi) > 1 + \chi\left(\frac{\gamma}{\sigma-1}-1\right) && \text{(IA.68)} \\
\iff\ &\gamma\theta > 1 + \frac{1}{\nu+\frac{1}{\gamma}+\frac{\sigma-2}{\sigma-1}}\left(\frac{1}{\sigma-1}-\frac{1}{\gamma}-1\right). && \text{(IA.69)}
\end{aligned}$$

As $n\to\infty$, we have $\widetilde{\mathcal{I}}_n\to\infty$ and $\mathcal{I}_1\to 0$, and therefore $\mathcal{I}_1/\widetilde{\mathcal{I}}_n\to 0$.

For the high-disagreement market-based spillover we have

$$spill_{mkt}(n\to\infty) = -(\sigma-1)\cdot\mathcal{E}_w + \mathcal{E}_{\mathcal{C}} = -\frac{\sigma-2}{\gamma}\cdot\left(\frac{1+\nu}{1+\nu+1/\gamma-1/(\sigma-1)}\right). \tag{IA.70}$$

When $\nu\to\infty$, there is a fixed supply of infrastructure and thus a fixed number of firms, which implies:

$$\begin{aligned}
spill_{out} &= \frac{1}{\sigma-1}, \\
spill_{mkt}(n\to\infty) &= -\frac{\sigma-2}{\gamma}.
\end{aligned}$$

E Additional Empirical Results

Table IA.1

Summary Statistics

	N	Mean	Std. Dev.	25th pct.	Median	75th pct.
Bubble						
Bubble Periods Dummy	2,734	0.0271	0.162	0	0	0
Value of Innovation (KPSS)						
Patent Level Value						
Stock Market ($ Mn)	1,171,806	14	37.8	2.32	5.46	12.7
Citations (fwd. looking)	1,171,806	12	22.6	2	5	13
Firm Level Value						
Stock Market ($ Mn)	47,887	232	1880	0.519	3.06	24.5
Citations (fwd. looking)	47,887	52.9	236	2.88	7.8	26.7
Firm Level Statistics						
Mkt. Cap. ($ Mn)	47,887	2854	14667	47.2	200	949
Segments (# NAICS-4)	53,066	1.33	0.646	1	1	2
Segments (# NAICS-6)	53,066	1.41	0.729	1	1	2
Measuring Spillovers (BSvR)						
Firm Outcomes (real or market valued)						
Sales ($ Mn)	9,382	3563	12626	135	509	2037
Tobin's q	9,382	2.46	3.09	0.86	1.49	2.71
Measures of Spillovers (Jaffe)						
Technology	9,382	9.8	1.02	9.34	9.97	10.5
Competition	9,382	7.32	2.35	6.33	7.64	9.01
Measures of Spillover (Mahalonobis)						
Technology	9,382	11.4	0.821	10.9	11.5	11.9
Competition	9,382	8.53	1.73	7.87	8.77	9.74

Note: Table IA.1 presents summary statistics of the main variables in the regression specifications. The bubble dummy corresponds to bubble detected across the Fama-French 49 industries, according to the methodology outlined in Greenwood, Shleifer, and You (2018). The value of innovation both at the firm and patent level is directly taken from Kogan et al. (2017). The stock market value of innovation at the patent level corresponds to the appreciation in the value of a firm issuing a patent around the patent-issuance date. The stock market value of innovation at the firm level corresponds to an annual aggregation of the total value of all patents issued by a firm in a given year. Both the patent- and firm-level citation value of a patent corresponds to its forward-looking number of citations (until the end of the sample in 2010). Other firm-level statistics correspond to the CRSP-Compustat merged file (for market capitalization, sales, and Tobin's q) and Compustat segments file. Tobin's q is measured from Bloom, Schankerman, and Van Reenen (2013) as the market value of equity plus debt divided by the stock of fixed capital. We use measures of technological and competition spillovers from Bloom, Schankerman, and Van Reenen (2013), corresponding to the distance between the technological class of patents issued by a firm with other public firms and to the distance between the set of product market of a firm and other public competitors.

Table IA.2

Number of Patents in Times of Bubbles

	Patents (#)	Log Patents (#)			
	(1)	(2)	(3)	(4)	(5)
Bubble	1.385**	0.148***	0.154**	0.169***	0.178**
	(0.578)	(0.056)	(0.066)	(0.063)	(0.075)
Lagged Patents (#)	0.982***				
	(0.026)				
Lagged Log Patents (#)		0.824***	0.795***	0.827***	0.799***
		(0.008)	(0.008)	(0.007)	(0.008)
Fixed Effects	C	–	C	Y	C, Y
Observations	106,176	106,278	106,176	106,278	106,176
R^2	0.91	0.67	0.68	0.68	0.68

Note: Table IA.2 presents panel regressions of the quantity of innovation, measured by the number of patents issued at the USPTO three-digit class level, on a dummy from Greenwood, Shleifer, and You (2018) that captures whether the firm is in an industry that is in a bubble or not. We control for the lagged number of patents for column one and lagged logarithm for columns two to five. Depending on the specification, we include fixed effects for the patent class level C and patent grant-year Y. Standard errors clustered at the year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

Table IA.3
Market Value of Patents Controlling for Volatility

	Panel A. Firm Volatility Controls					
	Patent Level			Firm Level		
	(1)	(2)	(3)	(4)	(5)	(6)
Bubble Dummy	0.332***	0.330***	0.201*	0.521***	0.428***	0.266**
	(0.090)	(0.090)	(0.120)	(0.116)	(0.122)	(0.133)
Log Citations (forward looking)		0.018***	0.015***		0.822***	0.820***
		(0.004)	(0.005)		(0.011)	(0.011)
Log Citations x bubble			0.055**			0.061***
			(0.025)			(0.012)
Firm-level Volatility	−0.312	−0.351	−0.341	−2.468**	−0.706	−0.699
	(0.798)	(0.794)	(0.798)	(1.125)	(1.026)	(1.029)
Fixed Effects	Y, F	Y, F	Y, F	Y, F	Y, F	Y, F
Observations	1,016,214	1,016,214	1,016,214	47,876	47,876	47,876
R^2	0.67	0.67	0.67	0.89	0.94	0.94

	Panel B. Industry Volatility Controls					
	Patent Level			Firm Level		
	(1)	(2)	(3)	(4)	(5)	(6)
Bubble Dummy	0.288***	0.287***	0.143	0.441***	0.350***	0.201**
	(0.079)	(0.078)	(0.105)	(0.064)	(0.070)	(0.088)
Log Citations (forward looking)		0.015***	0.012***		0.823***	0.821***
		(0.004)	(0.004)		(0.010)	(0.010)
Log Citations x bubble			0.061***			0.057***
			(0.018)			(0.013)
Industry-level Volatility	29.601***	29.501***	29.581***	24.488***	25.744***	25.651***
	(7.405)	(7.392)	(7.426)	(6.307)	(4.990)	(4.967)
Fixed Effects	Y, F	Y, F	Y, F	Y, F	Y, F	Y, F
Observations	1,089,416	1,089,416	1,089,416	47,886	47,886	47,886
R^2	0.69	0.69	0.69	0.89	0.94	0.94

Note: Table IA.3 presents panel regressions of the value of innovation, as measured in Kogan et al. (2017) at the patent and firm levels, on a dummy from Greenwood, Shleifer, and You (2018) that captures whether the firm is in an industry that is in a bubble or not. We control for firm-level volatility in panel A and for industry-level volatility in panel B (measured using daily returns on an annual basis). We include fixed effects for firm F and patent grant year Y. Standard errors clustered at the year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

Table IA.4

Sales, and Productivity at Different Horizons During Bubbles

	Horizon (years)				
	1	2	3	4	5
			Future Sales		
Citations	0.004	0.012***	0.017***	0.023***	0.028***
	(0.003)	(0.004)	(0.005)	(0.006)	(0.007)
Bubble	0.022	−0.026	−0.040	−0.047	−0.059
	(0.034)	(0.046)	(0.050)	(0.050)	(0.049)
Citations x Bubble	−0.003	0.004	−0.003	0.008	0.004
	(0.016)	(0.018)	(0.019)	(0.020)	(0.023)
Fixed Effects	Y, I	Y, I	Y, I	Y, I	Y, I
Observations	120,948	109,365	99,076	89,932	81,695
R^2	0.10	0.10	0.11	0.12	0.13
			Future TFPR		
Citations	0.007***	0.007***	0.009***	0.009***	0.010***
	(0.002)	(0.002)	(0.002)	(0.002)	(0.003)
Bubble	0.026**	0.019	0.012	0.001	−0.006
	(0.010)	(0.012)	(0.012)	(0.015)	(0.018)
Citations x Bubble	0.010	−0.005	−0.017	−0.001	−0.031
	(0.015)	(0.013)	(0.017)	(0.017)	(0.023)
Fixed Effects	Y, I	Y, I	Y, I	Y, I	Y, I
Observations	80,912	72,898	65,890	59,692	54,146
R^2	0.20	0.24	0.26	0.28	0.29

Note: Table IA.4 presents panel regressions of future sales and productivity (TFPR) for horizons of one to five years on a bubble dummy and a citation measure of patent value as in Kogan et al. (2017). We use industry (I) and year (Y) fixed effects. Standard errors clustered at the firm-year level are presented in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1

Table IA.5

Competitive Spillovers During Bubbles: Unnormalized Measures of Spillovers

	Market-based Spillovers			Outcome-based Spillovers		
	Jaffe	Mahalanobis	IV Jaffe	Jaffe	Mahalanobis	IV Jaffe
	(1)	(2)	(3)	(4)	(5)	(6)
Bubble x Spill-SIC	0.146***	0.211***	0.172***	0.004	0.003	0.004***
	(0.028)	(0.036)	(0.038)	(0.009)	(0.014)	(0.001)
Spill-SIC	−0.086***	−0.223***	−0.335***	−0.023***	−0.015	−0.046
	(0.018)	(0.044)	(0.101)	(0.005)	(0.015)	(0.044)
Spill-Tech	0.267**	0.854***	1.063***	0.121***	0.159***	0.157**
	(0.114)	(0.148)	(0.148)	(0.022)	(0.040)	(0.063)
Fixed Effects	Y, F	Y, F	Y, F	Y, F	Y, F	Y, F
Observations	8,910	8,946	8,910	8,789	8,825	8,789
R^2	0.74	0.74	0.73	0.99	0.99	0.99

Note: Table IA.5 presents panel regressions of firm value (log of sales or Tobin's q) on a measure of competition from Bloom, Schankerman, and Van Reenen (2013) interacted with a bubble dummy, measured as in Greenwood, Shleifer, and You (2018) that captures whether the firm is an industry that is in a bubble state or not. We control for the technological spillover measure that corresponds to the public firms that issue patent in similar technological space. We use a measure of spillovers based on the unnormalized Jaffe Covariance/Exposure Distance Metrics (see Table 8 of Bloom, Schankerman, and Van Reenen (2013)). We follow the specification from Table of 3 and 5 of Bloom, Schankerman, and Van Reenen (2013) and include the same controls. We use firm F and year Y fixed effects. Standard errors clustered at the year level are presented in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1%, respectively.

Table IA.6

Ownership and Bubbles: Concentration Ratio

	Panel A: Concentration ratio of portfolio in industry			
	(1)	(2)	(3)	(4)
Bubble	0.037***	0.022***	0.009*	0.005
	(0.008)	(0.005)	(0.005)	(0.004)
Controls	Yes	Yes	Yes	Yes
Fixed Effects	No	Industry	Date	Industry, Date
Observations	1,449,114	1,449,114	1,449,114	1,449,114
R^2	0.08	0.18	0.10	0.18

	Panel B: Concentration ratio of portfolio by institution type			
	(1)	(2)	(3)	(4)
Bubble x Banks	0.028*	0.020**	0.025***	0.020***
	(0.014)	(0.010)	(0.009)	(0.006)
Bubble x Insurance	0.051***	0.042***	0.039***	0.037***
	(0.010)	(0.007)	(0.008)	(0.005)
Bubble x Inv. Advisers	0.040***	0.021***	0.005	0.000
	(0.008)	(0.006)	(0.005)	(0.004)
Bubble x Mutual Funds	0.017***	0.001	−0.006	−0.013**
	(0.006)	(0.004)	(0.005)	(0.005)
Bubble x Pension Funds	0.037***	0.028***	0.016*	0.018***
	(0.012)	(0.007)	(0.009)	(0.005)
Controls	Yes	Yes	Yes	Yes
Fixed Effects	No	Industry	Date	Industry, Date
Observations	1,449,114	1,449,114	1,449,114	1,449,114
R^2	0.11	0.20	0.12	0.21

Note: Table IA.6 presents a regression of portfolio concentration within an industry on a bubble dummy in that industry-year. The bubble dummy corresponds to the first year where a bubble is detected across Fama-French 49 industries according to the methodology outlined in Greenwood, Shleifer, and You (2018). Industry portfolio concentration is measured using portfolio holdings the SEC 13F filings, where we construct the four-firm concentration ratio (CR4) of portfolio shares for a manager for each industry. In Panel B, manager types follow Koijen and Yogo (2019) and are derived from the SEC 13F filings. All specifications include controls for manager size, the number of stocks in a given industry, and industry volatility. Standard errors clustered at the date (quarterly) level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

Table IA.7

Ownership and Bubbles: Herfindahl Hirschman Index

	Panel A: Herfindahl of portfolio in industry			
	(1)	(2)	(3)	(4)
Bubble	0.022***	0.007**	0.007	−0.005
	(0.005)	(0.004)	(0.004)	(0.004)
Controls	Yes	Yes	Yes	Yes
Fixed Effects	No	Industry	Date	Industry, Date
Observations	1,449,114	1,449,114	1,449,114	1,449,114
R^2	0.06	0.13	0.06	0.14

	Panel B: Herfindahl of portfolio by institution type			
	(1)	(2)	(3)	(4)
Bubble x Banks	0.020*	0.009	0.022***	0.009
	(0.010)	(0.008)	(0.008)	(0.007)
Bubble x Insurance	0.026***	0.016***	0.023***	0.012*
	(0.007)	(0.005)	(0.006)	(0.006)
Bubble x Inv. Advisers	0.023***	0.007*	0.006	−0.007**
	(0.006)	(0.004)	(0.004)	(0.003)
Bubble x Mutual Funds	0.009**	−0.007**	−0.003	−0.018***
	(0.004)	(0.004)	(0.004)	(0.005)
Bubble x Pension Funds	0.012	0.003	0.002	−0.005
	(0.008)	(0.005)	(0.007)	(0.006)
Controls	Yes	Yes	Yes	Yes
Fixed Effects	No	Industry	Date	Industry, Date
Observations	1,449,114	1,449,114	1,449,114	1,449,114
R^2	0.08	0.15	0.08	0.16

Note: Table IA.7 presents a regression of portfolio concentration within an industry on a bubble dummy in that industry-year. The bubble dummy corresponds to the first year where a bubble is detected across Fama-French 49 industries according to the methodology outlined in Greenwood, Shleifer, and You (2018). Industry portfolio concentration is measured using portfolio holdings the SEC 13F filings, where we construct the Herfindahl-Hirschman index (HHI) of portfolio shares for a manager for each industry. In Panel B, manager types follow Koijen and Yogo (2019) and are derived from the SEC 13F filings. All specifications include controls for manager size, the number of stocks in a given industry, and industry volatility. Standard errors clustered at the date (quarterly) level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

Table IA.8

Diversity and Private Value of Innovation in Bubbles

	Patent Level			Firm Level		
	(1)	(2)	(3)	(4)	(5)	(6)
Bubble x Segments	−0.612***	−0.600***	−0.533***	−0.473***	−0.402***	−0.307***
(NAICS 4 digits)	(0.157)	(0.155)	(0.148)	(0.109)	(0.080)	(0.069)
Bubble	1.471***	1.441***	1.335***	1.576***	1.431***	1.180***
	(0.198)	(0.192)	(0.205)	(0.267)	(0.323)	(0.288)
Segments	0.309***	0.305***	0.296***	0.062	0.015	0.014
(NAICS 4 digits)	(0.097)	(0.096)	(0.100)	(0.043)	(0.045)	(0.037)
Log Citations		0.047***	0.044***		0.044***	0.044***
(forward looking)		(0.010)	(0.009)		(0.009)	(0.009)
Log Market Cap			0.154***			0.286***
(lagged)			(0.040)			(0.046)
Fixed Effects	Y, F	Y, F	Y, F	Y, F	Y, F	Y, F
Observations	180,636	180,636	177,911	10,426	10,426	10,256
R^2	0.72	0.72	0.72	0.88	0.93	0.94

Note: Table IA.8 presents panel regressions of the value of innovation, as measured in Kogan et al. (2017) at the patent and firm levels, on a bubble dummy interacted with the number of industries spanned by the different segments of the parent firm. The bubble dummy is from Greenwood, Shleifer, and You (2018) and captures whether the firm is in an industry that is in a bubble or not. Compustat segments are measured at the four-digit NAICS code level from the Compustat segments file. We control for the forward-looking number of citations generated by a patent (or firm) from Kogan et al. (2017), and the lagged market capitalization of the firm. We include fixed effects for firm F and patent grant year Y. Standard errors clustered at the firm-year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

Table IA.9
Diversity and Private Value of Innovation in Bubbles (NAICS 6 Digit Industries)

	Patent Level			Firm Level		
	(1)	(2)	(3)	(4)	(5)	(6)
Bubble x Segments (NAICS 6 digits)	−0.562*** (0.178)	−0.550*** (0.175)	−0.488*** (0.164)	−0.388** (0.155)	−0.370*** (0.094)	−0.295*** (0.072)
Bubble	1.456*** (0.232)	1.425*** (0.227)	1.329*** (0.235)	1.459*** (0.317)	1.389*** (0.342)	1.167*** (0.289)
Segments (NAICS 6 digits)	0.122 (0.096)	0.122 (0.095)	0.112 (0.101)	0.010 (0.044)	−0.031 (0.048)	−0.026 (0.041)
Log Citations (forward looking)		0.049*** (0.010)	0.047*** (0.009)		0.047*** (0.009)	0.047*** (0.009)
Log Market Cap (lagged)			0.156*** (0.042)			0.287*** (0.046)
Fixed Effects	Y, F	Y, F	Y, F	Y, F	Y, F	Y, F
Observations	180,636	180,636	177,911	10,426	10,426	10,256
R^2	0.71	0.71	0.72	0.88	0.93	0.94

Note: Table IA.9 presents panel regressions of the value of innovation, as measured in Kogan et al. (2017) at the patent and firm levels, on a bubble dummy interacted with the number of industries spanned by the different segments of the parent firm. The bubble dummy is from Greenwood, Shleifer, and You (2018) and captures whether the firm is in an industry that is in a bubble or not. Compustat segments are measured at the six-digit NAICS code level from the Compustat segments file. We control for the forward-looking number of citations generated by a patent (or firm) from Kogan et al. (2017) and the lagged market capitalization of the firm. We also include fixed effects for firm F and patent grant year Y. Standard errors clustered at the firm-year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

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