Bubbles and the Value of Innovation†

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July 3, 2020

Abstract

Episodes of booming innovation coincide with intense speculation in financial markets leading to bubbles—increases in market valuations and firm creation followed by a crash. We provide a framework reproducing these facts that makes a rich set of predictions on how speculation changes both the private and social values of innovation. We confirm the theory in the universe of U.S. patents issued from 1926 through 2010. Measures based on financial market information indicate that speculation increases the private value of innovation and reduces negative spillovers to competing firms. No commensurate change occurs in measures grounded in real outcomes.

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1 Introduction

Episodes of booming innovation often coincide with intense speculation in financial markets, leading to periods that have been described as bubbles: large increases in firm creation and market values followed by a crash (e.g. Scheinkman, 2014). This connection is natural: when faced with new ideas, people disagree; disagreement leads to bubbles in which high prices generate more investment. But a crucial aspect of how innovation spreads is missing from this narrative: the interactions of various actors and their spillovers on each other during the bubble.

We address this issue with a new theory that enriches the simple narrative in two ways, motivated by the following observation. To evaluate spillovers, each investor must have an opinion not only about the firms she invests in, but also about their competitors — how do other firms affect my firm and how do I affect others? First, we entertain consequential disagreement across agents about these questions, without succumbing to the high dimensionality they typically imply. In our model, investors disagree about which firms are more likely to succeed, in contrast with the existing literature's focus on disagreement about the aggregate state. Second, we account for a rich range of spillovers across firms, and show how they interact with speculation.

The theory delivers a set of novel insights about the private and social value of innovation during these episodes. We test the main predictions using over a million patents issued from 1926 through 2010. In both our model and the data, we find that during bubbles: *(i)* the increase in market value of a firm following the introduction of a new patent is larger; *(ii)* the relative spillover to the *market* value of competitors is smaller; and *(iii)* there is no commensurate change in *outcome-based* measures of value focusing on patent citations or output. In addition, the model highlights how the effect of bubbles depends on the source of spillovers—competition, learning, etc. We also show how these measures of spillovers can guide policy, emphasizing the importance of distinguishing between market- and outcome-based spillovers in the presence of speculation.

Formally, we analyze a model in which new ideas are implemented by firms that compete with each other. We allow for a rich range of firm interactions while remaining parsimonious enough to be tractable and yield clear testable predictions. In the first stage, firms are created by raising money on financial markets; in the second stage, competition and production occur. In our baseline analysis, competition takes a simple form: only a fixed number of the best firms get to produce. A novel ingredient of our theory is that investors agree to disagree about which firms will be more productive. Each investor picks her favorite firms for her portfolio and therefore values her investments more than her beliefs about the average firm.[1] This mechanism increases valuations and

[1]Van den Steen (2004) studies how disagreement and choice lead to optimism.

incentivizes more firms to enter. In the second stage, not all investors can be correct about which firms produce: prices drop and many firms fail. This rise and fall as a result of speculation echoes broad narratives of bubble episodes as well as some of their more subtle features. Disagreement is more natural in the face of new ideas, making these episodes more likely following the introduction of a broad new technology (Brunnermeier and Oehmke, 2013; Scheinkman, 2014) or among younger firms (Greenwood, Shleifer, and You, 2018). In a dynamic extension of the model, we show that they are also related to more intense trading activity (Hong and Stein, 2007; Greenwood, Shleifer, and You, 2018).

We use the model to study the private and social values of innovation. When people disagree, there appears to be a multitude of private and social values. Whose beliefs should one focus on? Our framework for heterogeneous beliefs overcomes this challenge. We are able to entertain rich disagreement and define private and social value corresponding to clearly measureable objects that all agents agree on. Market participants agree on the aggregate distribution of firms—and therefore all macroeconomic outcomes—but disagree on the relative positions of specific firms in this distribution. Each firm faces the same distribution of beliefs across investors even though the identity of these investors differs, leading to a symmetric equilibrium in the first stage. Using this symmetry, we define private and social value in both market-based and outcome-based ways. The market-based approach considers prices of firms in financial markets. Under this criterion, the private value of a firm is its price, while the social value of that firm is the amount by which its introduction changes the price of all firms in the economy.[2] From an outcome-based perspective, the private value is the average output of a firm, while the social value is how much the introduction of a firm changes total output in the economy.

The model makes two predictions about the market-based value of firms. First, it predicts that speculation increases the market-based private value of firms. When disagreement increases, investors have higher valuations of their chosen firms because they expect these firms to be more productive than the average firm in the economy, even though the aggregate productivity distribution of new firms is unchanged. Second, it predicts that the wedge between the market-based social and private values of a firm is attenuated by speculation. Each new firm introduces a *business-stealing effect* as it displaces other firms.[3] With speculation, investors believe they are investing in the most productive firms in the economy, making them less concerned about being displaced by a new firm. Their valuation of their chosen firms is thus impacted less by any additional firm entry. In contrast, the outcome-based measures of value remain unchanged since speculation affects neither the aggregate productivity distribu-

[2] In richer specifications, we also incorporate effects on other participants in the economy.

[3] While the term of business-stealing suggests malfeasance, it is simply the textbook name for how the entry of a new firm takes customers and profit away from existing firms, conveying a negative externality, see Acemoglu (2008).

tion nor the competitive structure.

A historical episode offers an illustration of our model. In 1686, William Phips secured funding in England from the Second Duke of Albemarle and his syndicate to search for sunken Spanish ships in the Bahamas.[4] He found 34 tons of treasure, yielding large returns to his investors. This successful expedition generated considerable speculation around treasure-hunting technology. The speculation was accompanied by a large spike in patenting activity for the time, with 17 patents for ways to recover underwater bounty registered between 1691 and 1693. Based on these innovations, numerous firms were introduced on equity markets and raised large amounts of capital despite competing for what was clearly a small pool of treasures. These high valuations were consistent with our notion of high market-based value that is unaffected by competitive spillovers in the presence of speculation. The boom was so large that it is sometimes credited for the emergence of developed equity markets in England. However, there was no corresponding increase in the outcome-based value of the new patents and firms because the expeditions only succeeded in finding a few worthless cannons.

To verify the model's predictions empirically, we use the universe of patents issued by public firms from 1926 through 2010 to measure the relation between innovation and market values. We follow the literature and construct empirical counterparts to the private and social values in our model. Specifically, we measure the market-based private value of new innovations using the stock returns of issuing companies in the days following a patent approval, as in Kogan, Papanikolaou, Seru, and Stoffman (2017). We estimate the market-based social value relative to private value of new innovations by using the response of firm valuations to innovations by their competitors, controlling for innovation by technologically related firms, as in Bloom, Schankerman, and Van Reenen (2013). We proxy for speculation by isolating bubble episodes following Greenwood, Shleifer, and You (2018): industry-years that have experienced a sharp run-up in stock prices.

During a bubble, the market-based private value of innovation increases by 30% at the patent level and between 40% to 50% at the firm level, corroborating the model's prediction that speculation increases private value. The effect is smaller for firms that span multiple industries. Intuitively, an investor is likely to have a range of views about the different products of a firm, hence reducing the effect of speculation about a given product on the firm's stock prices. The change in market-based private value is not accompanied by a corresponding change in outcome-based private value. In particular, we show that the increase in number of patent citations is small relative to the increase in market-based private value, reflecting little change in the quality of innovation in bubbles.

[4]See Scott (1912) for background information on this episode, a landmark in the history of the stock market in the United Kingdom.

The data also support the prediction that speculation dampens market-based but not outcome-based measures of spillovers. In particular, our estimates suggest that during bubbles the business-stealing effect measured by asset prices completely vanishes. In our model, this disappearance occurs asymptotically for large levels of speculation. We do not see any change in the business-stealing effect measured through sales during bubbles, again in line with our theory.

Competition is only one of the many ways through which the effects of an innovation are felt in the economy. An innovation also affects owners of scarce production inputs, alters consumers' choices over their consumption baskets, and allows other firms to learn. We incorporate these prominent sources of spillovers in our theory. Doing so is interesting for three reasons. First, we obtain new testable predictions of our theory for the interaction between speculation and innovation that can help discipline future empirical work on these channels. Second, by comparing across many different specifications, we obtain a more systematic typology of how speculation shapes spillovers. We find that the main determinant for the impact of speculation on market-based spillovers are the beliefs of those who *receive* the spillovers rather than those who *create* them. Third, we confirm that the divergence between market- and outcome-based measures is not only quantitative but also qualitative. The effect of industry characteristics on these two measures is often reversed. Speculation can also reverse their sign. Finally, while the outcome-based spillover only depends on macroeconomic quantities, the market-based one depends on the microeconomic structure of the economy.

Our results also have normative implications because the spillover measures are important inputs for optimal innovation policy. We derive the optimal Pigouvian tax on firm entry under different notions of social welfare. On the one hand, the Pareto approach is non-paternalistic: it evaluates the welfare of each agent under her own beliefs. Under this approach, the optimal entry tax is simply the opposite of the market-based measure of spillover since asset prices reflect agents' beliefs. On the other hand, a paternalistic approach evaluates welfare under a single belief, the population distribution. Then, the optimal tax combines two components: the outcome-based measure of spillover and the private-value distortion caused by the bubble . These two components might offset each other: by fostering more entry, a bubble might fix an economy that would otherwise have under-entry due to positive externalities. The two welfare approaches highlight that there is no single "correct" measure of spillover, but rather that different objectives call for different measures. Our theoretical results provide a mapping between outcome-based and market-based measures, allowing one to derive optimal tax policy using data on only one of the two types of measures.

Related Literature. Our approach to representing speculation and the bubbles that ensue builds on a large body of work. The effect of belief disagreement on asset prices in the theoretical literature goes back to Miller (1977), Harrison and Kreps (1978), and Scheinkman and Xiong (2003). More recent contributions include Barberis, Greenwood, Jin, and Shleifer (2018), who emphasize the role of extrapolation, and Chinco (2020), who incorporates social interactions. We enrich this literature by focusing on disagreement at the micro level, specifically across many firms. We develop a framework that makes the high-dimensionality of this problem tractable; we use it to consider implications for firm creation and innovation. Empirically, we use the methodology from Greenwood, Shleifer, and You (2018) to identify bubble episodes in stock market data. There is a broader set of empirical evidence that supports the relation between speculation, heterogeneous beliefs, and asset prices, such as Chen, Hong, and Stein (2002) or Diether, Malloy, and Scherbina (2002). Brunnermeier and Oehmke (2013) and Scheinkman (2014) review the theory and evidence.[5]

Our focus on the implications of bubbles for innovation connects us to work emphasizing the real effects of financial markets, dating back to Tobin (1969) and Hayashi (1982). Bond, Edmans, and Goldstein (2012) offer a survey. In the context of bubbles, Panageas (2015) highlights implications for investment, while Hombert and Matray (2020) focus on labor market outcomes.[6] Dong, Hirshleifer, and Teoh (2020) document effects of overvaluation on the quantity and ambitiousness of innovation at the firm level. We go beyond the direct effect of high stock prices by considering how speculation shapes the private and social values of innovation. To that end, we use data and measurement methods developed by Pakes (1986), Griliches, Pakes, and Hall (1986), Bloom, Schankerman, and Van Reenen (2013), and Kogan, Papanikolaou, Seru, and Stoffman (2017). Furthermore, our theoretical framework gives rise to a more nuanced set of tests of innovation transmission.

Naturally, there are other approaches to studying the relation between financial markets and innovation. Like us, this work builds on classic models of technological growth (e.g. Acemoglu, 2008). However, the extant literature considers different sources of asset-price fluctuation. One strand of the literature centers on uncertainty and risk compensation rather than speculation. Pastor and Veronesi (2005, 2009) offer quantitative accounts of innovation booms focused on learning. Gârleanu, Kogan, and Panageas (2012) and Kogan, Papanikolaou, and Stoffman (2020) highlight risks associated with displacement. Kung and Schmid (2015) study the long-run risk associated with innovation. Corhay, Kung, and Schmid (2015) and Loualiche (2020) measure risks associated with competition across firms. Another stream of work focuses on rational bubbles (as in Samuelson (1958) and Tirole (1985)) and whether they crowd in or crowd out investment. Farhi and Tirole (2011) and Martin and Ventura (2012) are two recent examples. Our results contribute to these lines of research by

[5] Janeway (2012) offers a first-person account of the interaction of bubbles and innovation.
[6] van Binsbergen and Opp (2019) quantify the effects of mispricing more generally.

highlighting important variations in the characteristics and social implications of innovation around speculative episodes.

Finally, we also contribute to the literature discussing welfare in the presence of disagreement. Heterogeneous beliefs raise the question of how to evaluate each agent's welfare.[7] One approach is to impose one common "correct" belief for welfare evaluation. Brunnermeier, Simsek, and Xiong (2014) circumvent the need to choose a specific belief by studying allocations that are efficient across all convex combinations of agents' beliefs. Davila (2020) follows this approach in the context of financial-transaction taxes. Kondor and Köszegi (2017) consider sophisticated financial products. A second approach uses the Pareto criterion and evaluates each agent's welfare under their own beliefs, which Duffie (2014) calls the "consenting-adults" criterion. We derive optimal innovation policies for both approaches and point out their differences. Interestingly, we relate the two types of policies to the measurement of both market prices and real outcomes.

The paper proceeds as follows. In Section 2, we introduce our model of speculation with business-stealing. In Section 3, we derive predictions for private and social values, which we verify empirically in Section 4. We extend the analysis to richer sources of firm interactions in Section 5 and consider the normative implications of our model in Section 6. Finally, Section 7 concludes.

2 A Model of Disagreement and Innovation

We introduce a stylized framework to study the interaction between speculation and innovation. Innovation and competition are represented by a set of firms implementing new ideas. Speculation is represented by households who disagree about which firms will succeed. Households trade shares of the firms. The interplay between these two aspects yields equilibrium patterns that are consistent with stylized facts about bubbles. We then use the model to make testable predictions about the private and social values of innovation.

Figure 1 illustrates the overall structure of the model. At date 0, households work to create blueprints. They sell the blueprints to firm creators, who implement them as firms. Households buy the claims to these firms' production. At date 1, firms compete and produce; households receive the payoffs from their positions in firms and consume. We now detail each of these steps.

[7] This question is closely related to the debate around the possibility that people agree to disagree; see Morris (1995) for a general perspective.

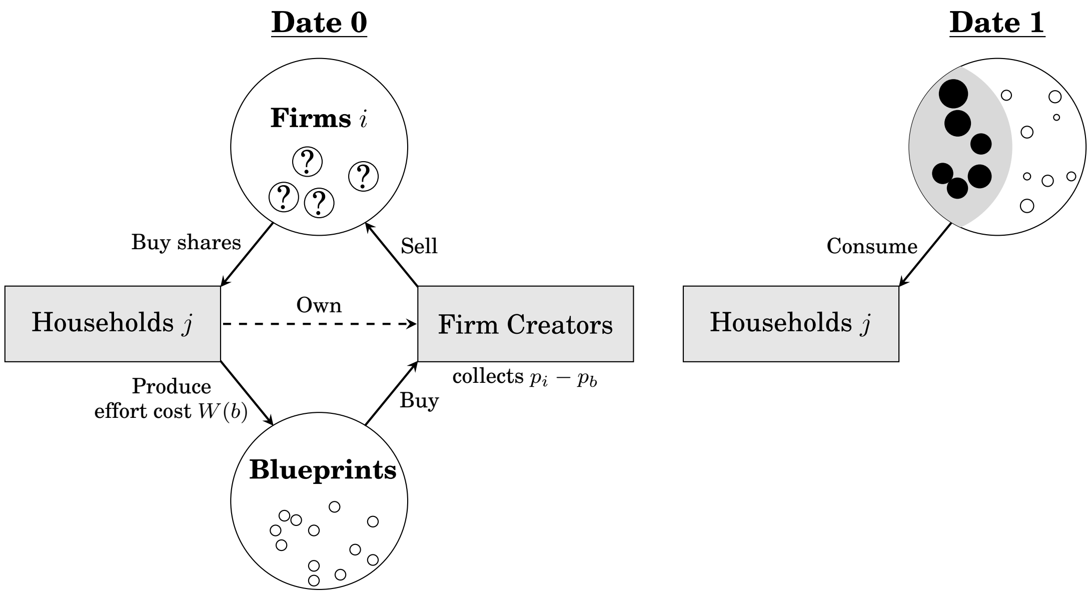

Figure 1
Summary of model structure.
The left panel represents the stage of firm creation at date $t = 0$, when blueprints are created and households buy shares in firms on financial markets based on their beliefs. The right panel represents the production stage at $t = 1$.

2.1 Firms

A continuum of firms, indexed by i and with total mass M_e, is created in equilibrium at date 0. At date 1, firms enter the production stage, and their productivity a_i is revealed. To capture competition across these firms, we assume that only a fixed number M of the most productive firms is able to produce. Given the cumulative distribution function of productivities in the population F, only firms above a cutoff $\underline{a}$ are able to produce, with

$$\underline{a} := F^{-1}\left(1 - \frac{M}{M_e}\right). \tag{1}$$

The profits of a firm with productivity a are then given by

$$\pi(a) = a^{\eta} \cdot \mathbf{1}\left\{a \geq \underline{a}\right\}, \tag{2}$$

where η determines how differences in productivities translate into differences in generated profits, and the indicator function captures whether the firm produces or not.[8] We concentrate on situations where $M_e < M$, so that fewer firms produce than are created; the case where $M_e = M$ is straightforward.

[8]The fact that the marginal active firm collects positive profits improves tractability but is not crucial to our conclusions. We show in Appendix C.3 that our results also hold in a variant of the model where the marginal firm earns zero profits.

This allocation of production slots is the key assumption to capture the notion of business-stealing. Indeed, firms do not internalize that they might take over the slot of another firm. The assumption of a fixed mass of production slots is especially plausible in industries that depend heavily on innovation. For instance, intellectual property law often provides exclusive use of a technology to its inventor.[9] We can interpret the fixed production slots as corresponding to a fixed number of M processes to produce the homogeneous good. The first firm to discover a process gets its exclusive use, and the speed of discovery is perfectly correlated with the productivity type a. Alternatively, we can assume that to produce, a firm needs one unit of an indivisible good that has not been discovered yet, and that only M of those exists in nature. Again firms with a higher type a find the ingredient faster.[10] Our assumption relies more generally on scarcity in the ability to produce that is not internalized by individual firms. We show in Appendix C how our results hold for a wide range of models of business-stealing, some of which allow for endogenous variation in M.

Formally, these firms constitute the whole economy. However, one can interpret the model as representing one sector of an economy.[11] Our results would be unchanged as long as firms do not interact across sectors because we assume the presence of a quasilinear good in the next section.

2.2 Households

Households play two roles in the model. They work to create the blueprints for new firms, and they speculate on which of the new firms will succeed. Formally, there is a unit mass of households indexed by j. At date 0, household j is endowed with a fixed unit of consumption good c_0 and her share of firm creators, which we describe in the next section. In addition, each household decides how many blueprints to supply, b_j. Blueprints are produced at a convex cost $W(b_j) = f_e b_j^{\theta+1} M^{-\theta}/(\theta+1)$, where the parameter θ is the elasticity of supply of blueprints and f_e controls the level of production costs. Finally, each household also decides on the number of shares to invest in each firm on the financial market, $\{s_i^j\}_i$. Households have heterogeneous beliefs about the distribution of productivity a_i for each firm i, which we describe in detail below. We assume

[9]Another motivation for the incompleteness is the difficulty of establishing markets for what has not been encountered yet. It is often the case that nobody owns something before it is discovered. For instance, how could we trade nuclear power before Henri Becquerel, Marie Curie, and Pierre Curie discovered radioactivity?

[10]Network goods or industries facing institutional constraints can face similar frictions in the allocations of productive positions that lead to a business-stealing effect—see Borjas and Doran (2012) for evidence in the context of scientific research.

[11]In an economy with multiple sectors, the inefficient levels of innovation we will show in the model are a form of misallocation.

that they can only take long positions in claims to firms.[12]

Households behave competitively and take prices as given. Hence, household j solves the problem

$$\max_{c_0, s_i^j \geq 0, b_j} \; c_0 + \mathbf{E}^j \left\{ \int s_i^j \pi_i di \right\} - W(b_j) \tag{3}$$

$$\textbf{s.t.} \;\; c_0 + \int s_i^j \; p_i \; di \leq 1 + p_b b_j + \Pi, \tag{4}$$

where p_i and p_b are the respective prices of firm i and blueprints, $\mathbf{E}^j$ is household j expectation and Π denotes firm creators' aggregate profits.[13]

2.3 Beliefs

To capture the notion of speculation, we assume that households disagree about which firms will be successful. Disagreement arises naturally in innovative episodes: households must rely on their priors to evaluate new firms (or ideas) and, in the absence of data on these firms, agree to disagree.[14] In general, it is challenging to keep track of heterogeneous beliefs about an entire set of firms. We overcome this issue by imposing structure on the distribution of beliefs. We assume that even though agents disagree about each individual firm, they agree on the population distribution of firm productivity.[15] We also assume that the population distribution F follows a Pareto distribution: $F(a) = 1 - a^{-\gamma}$ for $a \geq 1$.

A simple narrative for our specification of individual beliefs is as follows. Each household organizes firms into a continuum of packets containing n firms each and believes it knows the exact ranking of productivity draws within each

[12]Such a hard constraint facilitates the analysis, but the important assumption is some limit or cost to the ability to take short positions. The importance of limits to short-selling in models of disagreement has been well-known since the work of Miller (1977), Harrison and Kreps (1978), or more recently Scheinkman and Xiong (2003). D'avolio (2002) and Jones and Lamont (2002) document empirical evidence of such short-sale constraints, with the former showing that these constraints increase with disagreement. A theoretical justification is provided by Duffie, Garleanu, and Pedersen (2002), who microfound short-sale costs using a search model.

[13]The assumption of risk neutrality does not play any role in the analysis. Formally, all our results would be identical if the objective of households were: $c_0 + \mathcal{U}^{-1}\left(\mathbf{E}^j \left\{\mathcal{U}(\int s_i^j \pi_i di)\right\}\right) - W(b_j)$, where $\mathcal{U}(.)$ is an increasing and concave function.

[14]Morris (1995) discusses arguments for heterogeneous priors, many of which are particularly applicable in our setting. Besides the lack of information about new technology, the model generates specific testable predictions for the belief structure.

[15]A motivation for focusing on disagreement across firms rather than about the aggregate is the empirical observation that high firm entry in a sector often follows disruptive innovation, either through a large technological change or the introduction of new products. New firms then conduct micro innovations to take advantage of a macro innovation (Mokyr (1992)). Empirically, these episodes appear particularly relevant to economic growth, as discussed for instance in Abernathy (1978) and Freeman (1982).

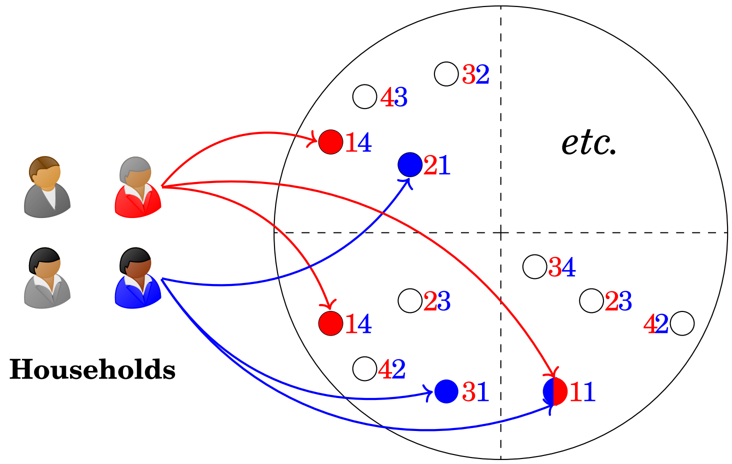

Figure 2
Households' beliefs and choice of firms to invest in.
The figure illustrates the case of $n = 4$. For this figure, the composition of each packet is identical across agents for exposition purposes. This choice does not affect the model results.

packet. We assume that the composition of packets and the order of firms within packets is drawn in an i.i.d. equiprobable fashion across agents and firms, and that each firm can only be in one packet. The parameter n controls the intensity of disagreement. When $n = 1$, households consider all firms to be the same, with their productivity drawn from F. As n increases, households can compare more firms in each packet and thus have a stronger prior that the best firm in each packet will have high productivity.

In equilibrium, household j only invests in the subset of firms that it considers to be the most productive in each of their respective packets. These firms are perceived by household j to have productivity drawn from F^n, the distribution of the maximum of n independent draws from F. Since households rank firms differently, they have different beliefs about the productivity distribution of any given firm and invest in different sets of firms, as illustrated in Figure 2. Each household believes that the firms it invests in are, in expectation, more productive than the average firm in the economy.

Our analysis does not rely on the specific narrative of packet formation, even though we find it intuitively appealing. The key mechanism is that when investors disagree, they each tilt their portfolio towards their favorite firms. As disagreement increases, this specialization strengthens, and investors in each given firm become more and more optimistic about its prospects. The crux of our assumptions is to reduce the distribution of beliefs to two parameters: the actual population distribution F and a single parameter n for the intensity of disagreement. Two features of our assumptions facilitate the analysis. First, while households disagree on which firms will succeed, they agree on the population distribution of firms. Hence, they agree on aggregate outcomes: both the threshold $\underline{a}$ and market conditions, which will play a role in the richer settings in Section 5. Second, beliefs are symmetric across households. This overcomes the issue of having to keep track of the entire distribution of beliefs and allocations.

2.4 Firm Creators

Finally, firm creators connect the step of innovation and trading in financial markets. They pay households to create blueprints, and issue claims to the corresponding firms on the financial market. Formally, there is a continuum of short-lived firm creators. At date 0, each firm creator can use a unit blueprint to create a new firm, which is then sold on competitive financial markets. They participate in competitive markets for blueprints and firms, taking their respective prices p_b and p_i as given.[16] The firm creator problem at time $t = 0$ is therefore:

$$\max_{c\in\{0,1\}} c \cdot (p_i - p_b). \tag{5}$$

2.5 Competitive Equilibrium

The competitive equilibrium of the economy is defined as follows. Firm creators maximize profits from selling their firms, taking prices as given. Households maximize their perceived expected utility by choosing their optimal blueprint discovery effort and a dynamic optimal portfolio allocation, taking the prices of blueprints and firms as given. Firms maximize profits given their production status. Finally, the markets for blueprints and claims to firms' profits (the stock market) clear:

$$\int b_j dj = M_e, \tag{6}$$

$$\forall i \in [0, M_e], \qquad \int s_i^j dj = 1. \tag{7}$$

Combining the equilibrium conditions yields a single equation determining the quantity of firm entry M_e:

$$W'(M_e) = V^{(n)}(M_e) = \int_{\underline{a}}^{\infty} \pi(a) dF^n(a). \tag{8}$$

In equilibrium, the marginal cost of creating an additional firm, $W'(M_e)$, is equal to the expected profits to an investor who favors it, $V^{(n)}(M_e)$. Both of these quantities are equal to the prices at which blueprints and firms trade in the

[16] We assume firm creators do not have any information about the firms they create.

economy, $p_i = p_b$. Therefore, this condition also pins down the price of new firms. We define the expected value of the firm:

$$\mathcal{I}_n(M_e, \eta) := \int_{F^{-1}(1-M/M_e)}^{\infty} a^{\eta} dF^n(a). \tag{9}$$

This value will play a prominent role in our analysis, with $V^{(n)}(M_e) = \mathcal{I}_n(M_e, \eta)$ in the model here.

2.6 Bubbles

Our framework yields equilibrium behavior consistent with features of bubble episodes related to the introduction of new technologies. Some well-documented examples include railroads, electricity, automobiles, radios, micro-electronics, personal computers, bio-technology, and the Internet. Scheinkman (2014) and Brunnermeier and Oehmke (2013) survey the field of speculation and bubbles; Janeway (2012) gives a first-person account of the relationship between innovation and speculation.

In the model, when investors disagree more, they assign a higher value to the firms they value most. As n increases, the demand for firms $V^n(.)$ shifts up. The higher demand from financial market participants increases the price of firms p_i and leads to more new firms M_e. This reflects the boom phase of a bubble: many new innovations are implemented and firms created, and prices on financial markets are high. Subsequently, however, not all investors can be right. Prices fall as output is below the level they imply. Formally, firms are priced at $\mathcal{I}_n$, under the distribution F^n, but the average output is only $\mathcal{I}_1$, under the population distribution F. This drop corresponds to the bust phase of a bubble. In the empirical analysis in Section 4, we use the abnormal price increase in the first phase of a boom to distinguish bubble periods, following Greenwood, Shleifer, and You (2018). While they also show that conditioning on a bust helps capture more features of bubbles, the ex-post nature of this conditioning would be an issue for our empirical tests.

A more formal definition of a bubble is a situation in which asset prices exceed an asset's fundamental value (see, for example, Brunnermeier (2016)). The drop in prices at date 1 is expected by all investors—they agree on aggregate outcomes—and supports this definition. How can all investors agree that the market portfolio is overpriced? Such a situation arises from households' heterogeneous beliefs. Different households view different firms as the most valuable and specialize their portfolios in those firms, which seem fairly priced to them. The short-sale constraint prevents each household from shorting the other firms, which it views as overpriced.[17] Chen, Hong, and Stein (2002), Diether, Malloy,

[17] Van den Steen (2004) describes how disagreement combined with optimal choice leads to overvaluation and Miller (1977) first pointed out the importance of short-sale constraints in financial markets.

and Scherbina (2002), and Yu (2011) establish empirically the link between dispersed beliefs and low future returns.[18]

Our model takes a static view of speculation, but in practice, investors' views change over time. In Appendix C.1, we study an extension of the model that entertains this possibility. A stronger form of overvaluation arises: the price of each firm exceeds the maximum valuation of its cash-flow by any specific investor in the economy. This difference comes from the fact that when investors change views, the current investor of a specific firm will typically not favor it anymore and sell it to somebody else. Hence, each time households trade firms signals a change in who values which firms the most, a mechanism similar to the models of Harrison and Kreps (1978) and Scheinkman and Xiong (2003). We show that this dynamic overvaluation is increasing in volume per period and the length of the bubble. Historically, abnormally high trading volume is seen as a hallmark of bubbles — see Scheinkman (2014) for a survey. For example, during the Roaring Twenties, daily records of share trading volume were reached ten times in 1928 and three times in 1929, with no new record set until 1968 (Hong and Stein, 2007). More recently, during the dot-com bubble, internet stocks had three times the turnover of otherwise similar stocks. Greenwood, Shleifer, and You (2018) also document that large stock price increases are more likely to end in a crash when they are accompanied by increased trading volume.

Finally, while our model does not provide an explicit premise for increases in disagreement, it is natural to expect disagreement during innovative episodes. As investors see these ideas and firms for the first time, they must rely on their priors to evaluate them. In contrast, more mature industries are likely associated with more common information that investors might have accumulated over time, accompanied by stronger agreement about what makes a firm successful. Consistent with this view, Greenwood, Shleifer, and You (2018) find that when the price run-up in an industry occurs disproportionately among the younger firms, crashes and low future returns are more likely.

All these observations suggest that the model is a useful representation of the episodes we are interested in. They are comforting as we turn to use the model to make new predictions about the value of innovation.

[18] Using investor survey data to quantify disagreement, Diether, Malloy, and Scherbina (2002) find that stocks with dispersed analysts' forecasts experience low subsequent returns. Yu (2011) aggregates this measure to portfolios such as the market portfolio and finds a similar result. Using stock market positions to measure disagreement, Chen, Hong, and Stein (2002) construct the fraction of the mutual fund population investing in a given stock, a measure of the breadth of ownership for individual stocks. They find that this measure predicts low stock returns.

3 The Value of Innovation

We are interested in two notions of the value of an innovation: the private value, which accrues to its investors, and the social value, which is its impact on the economy overall. Understanding these two notions is important because they shed light on how the process of innovation alters the economy and ultimately creates growth. We study two approaches to measure these values: *market-based* measures of value that use asset prices and *outcome-based* measures of value that use real outcomes, such as patent citations or output. In our model, these two spillover measures diverge in systematic ways during bubbles, providing predictions that we test in Section 4.

3.1 Private Value

The private value of a firm is the value of that firm to its investors. Consider first the market-based measure of this value. Empirically, the change in stock price of a firm following an innovation, as measured by Kogan et al. (2017), reveals the market-based private value. In the model, the market-based private value is simply the price $p_i = \mathcal{I}_n(M_e)$ at which the firm trades. This is because an innovation and a firm coincide, an assumption we make for simplicity.[19] As discussed in the previous section, this value increases during bubble episodes (large n).

The outcome-based counterpart to this metric is the actual effect of an innovation. In the data, it can be measured by the number of citations a patent receives or changes in sales following the introduction of an innovation. In the model, the realized output of the firm is the outcome-based private value. On average, the output of a firm is given by $\mathcal{I}_1(M_e)$, and all investors agree about this. In the absence of disagreement, $n = 1$, market-based and outcome-based private values coincide. However, as n increases, the two values diverge and the ratio $\mathcal{I}_n(M_e)/\mathcal{I}_1(M_e)$ increases. In other words, the increase in market value due to speculation does not reflect a commensurate increase in fundamentals.

These two predictions are the defining properties of bubbles applied to innovation. The market value of an innovation increases during a bubble. This increase is not justified by a change in outcomes. However, our analysis below of social value in the model paints a more subtle picture than a naive theory that simply states that all valuations are higher than usual during a bubble.

3.2 Social Value

The social value of a firm is the effect of introducing this firm on the whole economy. A new firm in the economy has a direct effect (its private value) and

[19]In our empirical exercise, we will come back to the distinction between innovation and firm.

an indirect effect because it affects the value of other firms. To characterize how social value departs from private value, we introduce the measure *spillover*. We define *spillover* as the indirect effect of a new firm scaled by its private value:

$$\textit{spillover} = \frac{\text{social value} - \text{private value}}{\text{private value}} \tag{10}$$

Empirically, this definition coincides with estimating the elasticity of firm value to the unexpected entry of another firm, closely related to Bloom, Schankerman, and Van Reenen (2013). Similar to the measure of private value introduced above, we consider both *market-based spillover* using asset prices and *outcome-based spillover* using output.

In the model with agreement, we find no distinction between the market- and outcome-based *spillover*, since the price of claims to a firm in period 0 is equal to the mean firm profits in period 1. In contrast, speculation introduces a gap between the two spillover measures, leading to novel empirical predictions.

3.2.1 Spillovers in the Model

In the model, spillovers arise because of the business-stealing effect. A new firm that enters may take up a production slot and displace an existing firm.

Computing the spillovers in the model is challenging because the baseline model of Section 2 does not feature unexpected entry: the number of firms is deterministic in equilibrium. We overcome this challenge by introducing the possibility that some blueprints randomly fail to be implemented. Formally, after firm creators purchase blueprints but before they introduce these blueprints to public markets, a fraction of them disappear. If M_e blueprints are created, either they all succeed or a mass Δ fails, with probability $1-\varepsilon$ and ε, respectively. By focusing on the limit when ε and Δ go to zero, we trace out the equilibrium effect of the unanticipated entry of an atomistic firm on the total value of the economy.

Comparing the total market value of the economy across the two outcomes, we obtain the market-based social value of an extra firm:

$$\begin{aligned} \textit{social value}_{mkt} &= \lim_{\Delta \to 0} \frac{M_e V^{(n)}(M_e) - (M_e - \Delta) V^{(n)}(M_e - \Delta)}{\Delta} && (11) \\ &= V^{(n)}(M_e) + M_e V^{(n)\prime}(M_e) && (12) \end{aligned}$$

For the outcome-based social value, simply replace n by 1 in this expression. We recognize the two effects of introducing a new firm. The first term is the direct effect: the private value of a new firm $V^{(n)}(M_e)$. The second term is the change in the value of all other firms in response to the entry of the new firm. Each of the M_e firms in the economy sees its value decrease by $V^{(n)\prime}(M_e)$. Taking the ratio of the direct and the indirect effect, we obtain the two measures of spillover:

$$spill_{mkt} = \frac{M_e V^{(n)\prime}(M_e)}{V^{(n)}(M_e)}, \tag{13}$$

$$spill_{out} = \frac{M_e V^{(1)\prime}(M_e)}{V^{(1)}(M_e)}. \tag{14}$$

We recognize the elasticity of the value of a firm to the number of firms, $\mathcal{E}_{\mathcal{I}_n}$.[20] Again, this elasticity is negative because of the business-stealing effect. The entry of a new firm makes all other firms less likely to produce.

In the remainder of this section, we derive properties of the spillover measures. In particular, we relate them to the intensity of speculation. Before doing so, it is worth pointing out a deeper interpretation of the spillover. The spillover measures the intensity of entry externalities in the economy and, as such, corresponds to a Pigouvian tax on entry. We come back to this implication at length in Section 6. For now, just note that for a non-paternalistic planner, the expression for the optimal tax rate is negative of the spillover: $\tau = -spill_{mkt}$.

3.2.2 Outcome-Based Spillover

The outcome-based spillover is the elasticity of the average output of a firm with respect to the number of firms in the economy. We compute explicitly $spill_{out}$ from the expression of equation (14), replacing the output of a firm with $\mathcal{I}_1$:

$$spill_{out} = -\frac{\displaystyle\int_{\underline{a}}^{\infty} \pi(\underline{a}) dF(a)}{\displaystyle\int_{\underline{a}}^{\infty} \pi(a) dF(a)} = -\frac{\gamma - \eta}{\gamma}. \tag{15}$$

The ratio of the two integrals has an intuitive explanation. The numerator represents the expected amount of profits displaced by the introduction of a new firm. It is the product of the output of the marginal producing firm (the firm that will get displaced) with the probability that that firm gets displaced, $\int_{\underline{a}}^{\infty} dF(a)$. The denominator is the expected output of a new firm.

Displacement lowers aggregate output, thus $spill_{out}$ is negative. However, displacement only occurs when the new firm is more productive than an existing firm, thus $|spill_{out}| < 1$. There are less spillovers if the distribution of firm productivity is more dispersed (lower values of γ) or when productivity differences translate into larger output differences (larger values of η).

The outcome-based spillover $spill_{out}$ does not depend on the equilibrium number of firms M_e. Technically, this result is the consequence of specifying a power

[20]Throughout the paper, we denote the elasticity of quantity X to firm entry M_e by $\mathcal{E}_X = d\log(X)/d\log(M_e)$.

production function and a Pareto productivity distribution.[21] As the level of disagreement changes, the outcome-based spillover stays constant. This prediction provides a useful benchmark against which to evaluate the behavior of market-based spillovers. In Section 4, we confirm this prediction empirically, thereby validating our framework.

3.2.3 Market-Based Spillover

The market-based spillover is the elasticity of the value of a firm to the number of firms in the economy. In the case of agreement ($n = 1$), market values and average outputs coincide; therefore, market-based and outcome-based measures of spillovers are identical. However, with disagreement, the two measures diverge, and $spill_{mkt}$ is not constant.

Proposition 1. *Speculation lowers the intensity of market-based spillovers:*

$$|spill_{mkt}(n)| < |spill_{mkt}(1)|, \quad \textit{for } n > 1 \tag{16}$$

Under disagreement, value spillovers are smaller despite the level of firm entry being higher. This seeming contradiction arises because the beliefs of households impact their valuation of equilibrium allocations. Under disagreement, households only invest in their favorite firms. Therefore, each household places a lower probability on being displaced by new entrants, thereby reducing the effect of the business-stealing externality on the market prices that determine the market-based spillovers.

More formally, we can rewrite the market-based spillover from (13) in its integral form:

$$spill_{mkt}(n) = \frac{M_e V^{(n)\prime}(M_e)}{V^{(n)}(M_e)} = -\frac{\displaystyle\int_{\underline{a}}^{\infty} \pi(\underline{a}) \frac{F_n'}{F'}(\underline{a})\, dF(a)}{\displaystyle\int_{\underline{a}}^{\infty} \pi(a) \frac{F_n'}{F'}(a)\, dF(a)}. \tag{17}$$

This ratio compares the value of displaced firms to the value of the firm displacing them, evaluated through the beliefs of their respective investors. Holding M_e and thus $\underline{a}$ constant, consider how this ratio changes with n. The numerator is the expected value of a displaced firm from the point of view of its owners. Disagreement affects this value through the change in probability weights F_n'/F' *at the threshold* $\underline{a}$. In contrast, the denominator is the profit of an average firm from the point of view of its owners. This quantity is affected by changes in the probability weights throughout the distribution *above the threshold*. Because increasing n corresponds to shifting the perceived distribution of productivities to

[21] Under a Pareto distribution, the ratio between marginal and average productivity is independent of the lower cutoff.

the right, the change in probability weights is increasing as we move to higher productivities (see Appendix Figure IA.1). Such an increase affects expected profits more strongly than the value of displaced firms, decreasing the wedge. In Appendix A.1, we show that this result holds more generally with minimal assumptions on the productivity distribution $F(\cdot)$ and profit function $\pi(\cdot)$.

This proposition gives rise to a clear prediction we test in the data: the market-based spillovers are lower in speculative periods. Following Bloom, Schankerman, and Van Reenen (2013), we can measure these spillovers by gauging the reaction of firms' valuations to variation in the amount of innovation by their competitors. In addition, we can also compare market-based and outcome-based spillovers. An immediate corollary of Proposition 1 is that the presence of speculation reduces market-based spillovers relative to outcome-based spillovers. These results of our model of bubbles differ from naive theories. For example, one common view is that bubbles inflate the price of all firms proportionally, either because people use too low a discount rate or overestimate the output of all firms. Under such a theory, because the market-based spillover is a ratio of valuations, it is unaffected and remains equal to the output-based spillover.

High speculation limit. We now consider the limiting case of high speculation: $n \to \infty$. This case is extreme: the total quantity of entry goes to infinity. However, it is useful because it gives rise to sharp characterizations of the spillovers, allowing us to highlight the distinction between market-based and outcome-based spillovers with disagreement.

Proposition 2. *In the high-disagreement limit ($n \to \infty$), the market-based spillover converges to a finite limit that depends on the sign of $\gamma\theta - \eta$:*

- *If $\gamma\theta > \eta$, the market-based spillover vanishes:* $\lim_{n\to\infty} spill_{mkt}(n) = 0$
- *If $\gamma\theta < \eta$, then τ converges to the spillover in the agreement case ($n = 1$):* $\lim_{n\to\infty} spill_{mkt}(n) = \frac{\gamma-\eta}{\gamma}$
- *In the knife-edge case of $\gamma\theta = \eta$,* $\lim_{n\to\infty} spill_{mkt}(n) = \check{spill}_{mkt} > -\frac{\gamma-\eta}{\gamma}$*, where $\check{spill}_{mkt}$ is defined in Appendix equation* (IA.16).

Figure 3 illustrates these cases. Appendix C.2 shows that the results hold in more general models of business stealing.

Two forces determine the asymptotic behavior of the market-based spillover. First, with more disagreement, investors increasingly believe that the firms they invest in are in the right tail of the productivity distribution. They are therefore less concerned about the risk of being displaced by new entrants because they expect that a smaller mass of firms in their portfolio will fail to meet the entry

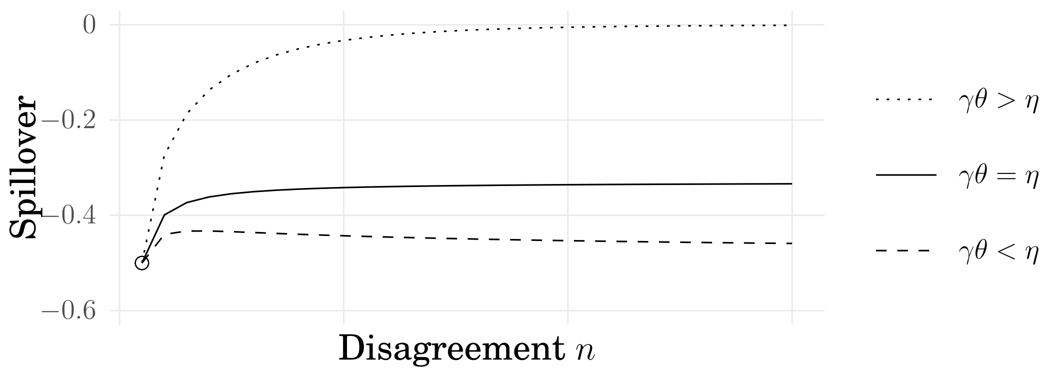

Figure 3
Market-based spillover with increasing disagreement.

threshold. For a given level of entry M_e, this mass converges to 0 as n goes to infinity. This is an extreme case of the result in Proposition 1.

Second, disagreement increases firm entry. For a given level of disagreement, n, as M_e converges to infinity, the M producing firms end up in the tail of both the population distribution, F, and the favorite-firm distribution, F_n. The tails of these two distributions have the same shape since $\lim_{x\to\infty} F_n'(x)/F'(x) = n$.[22] Therefore, disagreement does not affect the relative position of the marginal and average valuation in the tail. This force brings the market-based spillovers back toward the level of outcome-based spillovers.

The relative strength of the forces depends of how fast firm creation increases with speculation. If $\gamma\theta > \eta$, the first force dominates. When θ is large, the marginal cost of firm creation rises more rapidly, which reduces equilibrium entry M_e and weakens the second force. As γ increases, we have a thinner-tailed firm-productivity distribution. The size of the tail becomes less important than the relative ordering of firms, which weakens the second force. As η decreases, profits increase less with productivity, which again diminishes the importance of the second force. In the data, we find that the valuations increase faster than quantity during bubbles, consistent with the case of $\gamma\theta > \eta$.[23] The relevant prediction is therefore a decrease in spillovers toward 0.

[22]This extreme value theory result is not the byproduct of power distributions but rather applies to a larger class of distributions.

[23]Specifically, we find in the next section that the valuation of innovation increases by 40% during bubbles, while the quantity of innovation increases only by 15%, so $\theta > 1$. Because $\eta < \gamma$ is always satisfied, we have $\gamma\theta > \eta$.

4 Empirical Evidence

We now take the three main predictions of the model to the data. First, the market-based measure of the private value of innovation increases with speculation. Second, market-based competitive spillovers decrease in the presence of speculation. Finally, outcome-based measures respond relatively less, or not at all.

4.1 Data

To test these predictions, we combine variation in whether an industry experiences a bubble with measures of the private and social values of innovation at the firm and patent level. This approach maps to an interpretation of the model in which there are multiple sectors, and each sector and time experiences its own level of disagreement. Summary statistics of the main regression variables are contained in Table 1.

Bubbles. We proxy for the presence of speculation using the empirical definition of bubbles from Greenwood, Shleifer, and You (2018). We split firms into 49 industries following the classification from Fama and French (1997). An industry-month is defined as being in a bubble if it satisfies three conditions simultaneously. First, the value-weighted portfolio of the corresponding industry experienced a return of 100% or more over the previous two years. Second, this industry value-weighted return also exceeds the return of the market by at least 100% over the past two years. Third, the industry value-weighted return over the past five years is larger than 50%. We aggregate to the industry-year level—the coarseness of the remainder of our data—by considering an industry-year in a bubble if the industry is in a bubble for at least a month in the year. In practice, multiple months are always in a bubble year due to the persistence of the bubble criteria. This approach identifies 74 industry-years in a bubble between 1962 and 2017.

Consistent with our model, Greenwood, Shleifer, and You (2018) find that "price run-ups ... involving younger firms [and] having higher relative returns among the younger firms ... [are] more likely to crash." These industries with young and innovative firms are likely to have scarce information, making them more susceptible to the disagreement that we model. We also confirm the validity of the bubble classification empirically and show that it predicts increases in innovation like the model. In Appendix Table IA.1, we show that there are 1.4 more patents issued within a USPTO technology class during bubbles, which translates to a 15% increase in the number of patents created in a patent class during a bubble.

Table 1
Summary Statistics

	N	Mean	Std. Dev.	25th pct.	Median	75th pct.
Bubble						
Bubble Periods Dummy	2,734	0.0271	0.162	0	0	0
Value of Innovation (KPSS)						
Patent Level Value						
Stock Market ($ Mn)	1,171,806	14	37.8	2.32	5.46	12.7
Citations (fwd. looking)	1,171,806	12	22.6	2	5	13
Firm Level Value						
Stock Market ($ Mn)	47,887	232	1880	0.519	3.06	24.5
Citations (fwd. looking)	47,887	52.9	236	2.88	7.8	26.7
Firm Level Statistics						
Mkt. Cap. ($ Mn)	47,887	2854	14667	47.2	200	949
Segments (# NAICS-4)	53,066	1.33	0.646	1	1	2
Segments (# NAICS-6)	53,066	1.41	0.729	1	1	2
Measuring Spillovers (BSvR)						
Firm Outcomes (real or market valued)						
Sales ($ Mn)	9,382	3563	12626	135	509	2037
Tobin's q	9,382	2.46	3.09	0.86	1.49	2.71
Measures of Spillovers (Jaffe)						
Technology	9,382	9.8	1.02	9.34	9.97	10.5
Competition	9,382	7.32	2.35	6.33	7.64	9.01
Measures of Spillover (Mahalonobis)						
Technology	9,382	11.4	0.821	10.9	11.5	11.9
Competition	9,382	8.53	1.73	7.87	8.77	9.74

Note: Table 1 presents summary statistics of the main variables included in the regression specifications. The bubble dummy corresponds to bubble detected across the Fama-French 49 industries, according to the methodology outlined in Greenwood, Shleifer, and You (2018). The value of innovation both at the firm and patent level is directly taken from Kogan et al. (2017). The stock market value of innovation at the patent level corresponds to the appreciation in the value of a firm issuing a patent around the patent-issuance date. The stock market value of innovation at the firm level corresponds to an annual aggregation of the total value of all patents issued by a firm in a given year. Both the patent- and firm-level citation value of a patent corresponds to its forward looking number of citations (until the end of the sample in 2010). Other firm-level statistics correspond to the CRSP-Compustat merged file (for market capitalization, sales, and Tobin's q) and to the Compustat segments file. Tobin's q is measured from Bloom, Schankerman, and Van Reenen (2013) as the market value of equity plus debt divided by the stock of fixed capital. We obtain measures of technological and competition spillovers from Bloom, Schankerman, and Van Reenen (2013), corresponding to the distance between the technological class of patents issued by a firm with other public firms and to the distance between the set of product market of a firm and other public competitors.

Private value of innovation. We use the measures of private value of innovation from Kogan et al. (2017). Their dataset combines stock market and patent data for U.S. firms for the period from 1926 through 2010. They measure the stock market response in the three-day window after a firm has a new patent issued, controlling for the return on the market portfolio during that period. This number is the direct counterpart to the market-based measure of private value of a new innovation in our framework. Kogan et al. (2017) also aggregate the stock market value from all the patents of a given firm every year to measure the value of innovation at the firm level. To capture the ultimate quality of each patent, Kogan et al. (2017) use the forward-looking number of citations generated by a patent for their patent-level analysis and the number of citations generated by all the patents produced by a firm in a given year for their firm-level analysis. This corresponds to the outcome-based measure of the private value of a new innovation in the model. While our baseline model takes each firm to correspond to exactly one blueprint and thus does not distinguish between the patent and firm level, we conduct both patent-level and firm-level empirical analysis and show that the two approaches support our model predictions.

Social value of innovation. For the social value of innovation, Bloom, Schankerman, and Van Reenen (2013) identify spillovers from different sources of firm interactions. They regress a firm-level outcome—log market value or log future sales—on the quantity of innovation by groups of "neighboring" firms. Because the set of close competitors and the set of close innovators do not coincide, this approach separates competitive spillovers arising from firms in neighboring industries and knowledge spillovers coming from firms issuing patents in the same technology USPTO class. Specifically, they first construct distances between firms in each of these two spaces. Then, for each firm-year, they compute distance-weighted stocks of innovative capital from all other firms. The resulting firm-level exposures are *spillsic* and *spilltech*, respectively, and regressions on these quantities measure competitive and innovative spillovers. Because our baseline model only considers competitive interactions—Section 5 extends to other sources of interactions—we are particularly interested in competitive spillovers. Nevertheless, we control for innovative interactions. When the left-hand side of the regression is log market value, this approach identifies the market-based spillover. When focusing on log sales, the coefficients identify the outcome-based spillover. Taking the log ensures that we are considering a semi-elasticity, the counterpart to the spillover in the model.

Bloom, Schankerman, and Van Reenen (2013) propose a variety of metrics for distance. We follow their construction. Proximity between two competing firms in the product market space is the correlation of the firms' distribution of sales across their industry segments. Technology proximity is analogously defined as the correlation of patent USPTO technology classes between firms, following earlier work by Jaffe (1986). The quantities of innovation by competitors and close innovators are the stocks of innovation weighted by these correlations.

The stock of innovation is constructed using a perpetual-inventory approach. Both measures are extended using the Mahalanobis distance to allow for flexible weighting of the correlation between firms across different technology or product market classes. Further details on the measures of spillovers can be found in Appendix E and in Bloom, Schankerman, and Van Reenen (2013).

4.2 The Private Value of Innovation During Bubbles

Market-based private value of innovation increases in bubbles. To study the effect of speculation on the private value of innovation at the patent level, we consider the following regression specification:

$$\log \xi_{j,t} = \beta B_{j,t} + \gamma Z_{j,t} + \varepsilon_{j,t}, \tag{18}$$

where $\xi_{j,t}$ is the private value of patent j issued during year t using the market-based measure from Kogan et al. (2017). The variable $B_{j,t}$ is an indicator for whether the firm issuing patent j was in an industry experiencing a bubble during year t. As in Kogan et al. (2017), the controls $Z_{j,t}$ include log of the number of citations for the patent, market capitalization, and year dummies. We take the market capitalization lagged by one year because the value of a firm's stock tends to rise during a bubble. The lagged market capitalization variable controls for the size of the firm without contaminating our estimate of how the private value varies with speculation. To ensure that our results are not driven by the type of firms or industries that go through bubbles, we include firm fixed effects. We run a similar regression at the firm level, replacing the private value and citation variables with their firm-level analogs. We no longer control for market capitalization because the firm-level measure of private innovation value is normalized by book value. Following Kogan et al. (2017), we include year fixed effects in this specification.

Table 2 shows that speculation indeed increases the private value of innovation. In particular, the presence of a bubble increases the private value of innovation by approximately 30% at the patent level and 40% to 50% at the firm level. These effects are both economically and statistically significant.

We stress that an increase in the market-based measure of the private value of innovation in a bubble is not mechanical. We measure bubbles as times of high stock market valuations, which do not necessarily coincide with a strong positive response to the news of issuing a patent. This positive response is a direct implication of our model in which bubbles arise in times of high private value of innovations. In contrast, the common view that bubbles are episodes in which prices are completely disconnected from fundamentals would not make such a prediction. Under this view, valuations could be high overall but would not be responsive to patent issuance.

Table 2
Private Value of Innovation in Bubbles

	Patent Level			Firm Level		
	(1)	(2)	(3)	(4)	(5)	(6)
Bubble Dummy	0.317***	0.315***	0.306***	0.514***	0.427***	0.427***
	(0.094)	(0.094)	(0.096)	(0.114)	(0.123)	(0.080)
Log Citations		0.016***	0.023***		0.823***	0.716***
(forward looking)		(0.004)	(0.005)		(0.011)	(0.009)
Log Market Cap			0.562***			0.626***
(lagged)			(0.028)			(0.020)
Fixed Effects	Y, F	Y, F	Y, F	Y, F	Y, F	Y, F
Observations	1,171,806	1,171,806	1,169,860	47,886	47,886	47,484
R^2	0.68	0.68	0.74	0.89	0.94	0.96

Note: Table 2 presents panel regressions of the value of innovation, as measured in Kogan et al. (2017) at the patent and firm levels, on a dummy from Greenwood, Shleifer, and You (2018) that captures whether the firm is in an industry that is in a bubble or not. We control for the lagged market capitalization of the firm and include fixed effects for firm F and patent grant year Y. Standard errors clustered at the grant-year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

Diversity dampens the effect of speculation. In our model, one firm coincides with a single blueprint. In the data, firms can issue multiple patents during the same year. The empirical distinction between patent and firm yields an additional source of variation not captured by our simple theory: some firms operate in multiple sectors. This variation naturally interacts with our mechanism: the effect of a bubble should be smaller for firms that have more diversified activities. Narrow firms, as opposed to firms that span multiple product lines, cater to specific investors who tend to have strong beliefs in their success. Thus, we expect a "conglomerate discount" on the value of innovation in times of bubbles: in the presence of disagreement or bubbles, multiple-product firms experience a smaller increase in the value of their innovation than narrow firms.[24]

To test this prediction empirically, we measure diversity with the variable $Segments_{j,t}$, which is the number of four-digit NAICS industries that a firm is active in. We collect this information from the Compustat segments files. We augment regression (18) with $B_{j,t} \times Segments_{j,t}$.[25]

As predicted, we find in Table 3 that the greater the number of segments, the less the private value of innovation increases in a bubble, as indicated by the

[24]Reed, Saffi, and Van Wesep (2020) provide empirical evidence of a similar conglomerate discount in stock valuations. Huang et al. (2020) document a similar discount in portfolios.

[25]Appendix Table IA.3 shows that the results are robust to using a definition of different segments using six-digit NAICS industries.

Table 3
Diversity and Private Value of Innovation in Bubbles

	Patent Level			Firm Level		
	(1)	(2)	(3)	(4)	(5)	(6)
Bubble x Segments (NAICS 4 digits)	-0.612^{***} (0.157)	-0.600^{***} (0.155)	-0.533^{***} (0.148)	-0.473^{***} (0.109)	-0.402^{***} (0.080)	-0.307^{***} (0.069)
Bubble	1.471^{***} (0.198)	1.441^{***} (0.192)	1.335^{***} (0.205)	1.576^{***} (0.267)	1.431^{***} (0.323)	1.180^{***} (0.288)
Segments (NAICS 4 digits)	0.309^{***} (0.097)	0.305^{***} (0.096)	0.296^{***} (0.100)	0.062 (0.043)	0.015 (0.045)	0.014 (0.037)
Log Citations (forward looking)		0.047^{***} (0.010)	0.044^{***} (0.009)		0.044^{***} (0.009)	0.044^{***} (0.009)
Log Market Cap (lagged)			0.154^{***} (0.040)			0.286^{***} (0.046)
Fixed Effects	Y, F	Y, F	Y, F	Y, F	Y, F	Y, F
Observations	180,636	180,636	177,911	10,426	10,426	10,256
R^2	0.72	0.72	0.72	0.88	0.93	0.94

Note: Table 3 presents panel regressions of the value of innovation, as measured in Kogan et al. (2017) at the patent and firm levels, on a bubble dummy interacted with the number of industries spanned by the different segments of the parent firm. The bubble dummy is from Greenwood, Shleifer, and You (2018) and captures whether the firm is in an industry that is in a bubble or not. Compustat segments are measured at the four-digit NAICS code level from the Compustat segments file. We control for the forward-looking number of citations generated by a patent (or firm) from Kogan et al. (2017), and the lagged market capitalization of the firm. We include fixed effects for firm F and patent grant year Y. Standard errors clustered at the grant-year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

significantly negative coefficient on the $B_{j,t} \times Segments_{j,t}$ regressor. Moreover, the effect of a bubble on the value of a patent for a firm with more than one segment is not statistically significant. Qualitatively, we find the same results whether we run the regressions at the patent level or the firm level.

Innovation quality does not see a commensurate increase in bubbles. Another prediction of our theory is that the market-based private value of innovation increases during bubbles *relative* to the outcome-based private value. In the model, this comparison is particularly sharp because firm productivity is drawn from the same distribution F in date 1 regardless of the level of disagreement, thus innovation quality does not depend on speculation. We now show empirically that there is, in fact, no significant increase in innovation quality, the outcome-based measure of the value of innovation, during bubbles.

In particular, we consider:

$$\log (1 + C_{j,t}) = \beta B_{j,t} + \gamma Z_{j,t} + \varepsilon_{j,t}, \tag{19}$$

Table 4
Number of Citations for Innovation in Bubbles

	Patent Level		Firm Level	
	(1)	(2)	(3)	(4)
Bubble Dummy	0.112***	0.113***	0.106***	0.107***
	(0.028)	(0.027)	(0.034)	(0.041)
Log Market Cap		−0.034***		0.234***
(lagged)		(0.008)		(0.008)
Fixed Effects	Y, F	Y, F	Y, F	Y, F
Observations	1,171,806	1,169,860	47,887	47,484
R^2	0.35	0.35	0.78	0.79

Note: Table 4 presents panel regressions of the number of citations (forward looking until the end of the sample), as measured in Kogan et al. (2017) at the patent and firm levels, on a bubble dummy interacted with the number of industries spanned by the different segments of the parent firm. The bubble dummy is measured from Greenwood, Shleifer, and You (2018) and captures whether the firm is in an industry that is in a bubble state or not. We control for the lagged market capitalization of the firm and include fixed effects for firm F and patent grant year Y. Standard errors clustered at the grant-year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

where $C_{j,t}$ is the forward-looking number of citations received by a patent j issued in year t. The controls $Z_{j,t}$ include the lagged market value of the firm, and a year and firm fixed effect as in equation (18). As before, we also run the firm-level analog.

Table 4 reports that the quality of innovation sees a slight increase during bubbles. The number of citations is 10% higher in times of bubbles. This increase arises partly from the higher number of patents issued during bubbles.

To understand why the increase in citations is small relative to the increase in the market value of innovation in bubbles from Table 2, we need to translate these two sets of estimates into the same units. We convert citations into dollars by reproducing the estimates from Kogan et al. (2017) in Appendix Table IA.4. We find that within a patent-class-year (i.e. excluding the variation due to bubbles), a 10% increase in citations corresponds to a 0.1% to 1.7% increase in stock market valuation. This number is substantially smaller than the direct effect of a 30% increase in market-based private value during bubbles. Thus, the quality of innovation as measured by future citations does not experience a rise comparable to the market-based private value, in line with the model predictions.

4.3 The Social Value of Innovation During Bubbles

Our model predicts that market-based measures of spillovers from competitors are dampened during bubbles, while outcome-based measures of spillovers are not. We test these hypotheses empirically by enriching the specification of Bloom, Schankerman, and Van Reenen (2013) to estimate spillovers conditional on a bubble:

$$\begin{aligned}\log X_{i,t} = {} & \beta\left(B_{i,t} \times \log spillsic_{i,t}\right) + \gamma_1 \log spillsic_{i,t} \\ & + \gamma_2 \log spilltech_{i,t} + \gamma_3 B_{i,t} + \delta Z_{i,t} + \varepsilon_{i,t},\end{aligned} \tag{20}$$

where $X_{i,t}$ is either the market value (Tobin's q) or output (sales normalized by an industry price index) of firm i in year t. As before, $B_{i,t}$ is an indicator of whether firm i is in an industry that experienced a bubble in year t. The controls $Z_{i,t}$ are taken from Bloom, Schankerman, and Van Reenen (2013) and include firm and year fixed effects. Taking $X_{i,t}$ to be the market value of the firm, we measure the market-based spillovers. On the other hand, for outcome-based measures of spillovers, we take $X_{i,t}$ to be firm output. Our hypotheses are that β is positive for the market value, and β is equal to 0 for firm output. Market-based spillovers disappear with speculation as investors ignore the business-stealing effect, while outcome-based spillovers are unchanged.

Table 5 shows that the regression results are consistent with the predictions of our model. The coefficient on the interaction term captures the change in spillovers accompanying an increase in speculation. The coefficient is significantly positive in the market-based spillover regressions, indicating a reduction in the business-stealing effect. Moreover, the estimated business-stealing effect in a bubble, captured by the sum of coefficients on $spillsic$ and the interaction term, is not significantly different from zero, although the point estimate is positive. These estimates suggest that the business-stealing effect vanishes completely during bubbles, consistent with our high speculation asymptotics in Proposition 2 with inelastic entry ($\gamma\theta > \eta$). In contrast, the presence of a bubble does not have any effect on outcome-based measures of spillovers. Again, this result is in line with our model.

Besides confirming our theory, these results also reject some alternative theories of bubble episodes. Common views of the episodes we assign as bubbles are that these are periods of high expectations of fundamentals (rational or not) or of low discount rates. These theories predict a proportional increase in all market valuations. Recalling that spillovers are the ratio of valuations of displaced profits from new innovation, these alternative theories pose two conflicts with our findings. First, they predict that market-based spillovers would tend to be unchanged. Second, if market-based spillovers vary due to changes in the nature of innovation, they would do so in parallel with the outcome-based spillovers.

These results highlight the importance of distinguishing between market-based and outcome-based measures of spillovers and, more generally, of the value of innovation during speculative episodes. While we have focused on competition as the source of spillovers thus far, we now show how to enrich our theory to make further predictions specific to other sources of spillovers. Test-

Table 5
Social Value of Innovation and Bubbles

	Market-based Spillovers		Outcome Spillovers	
	Jaffe (1)	Mahalanobis (2)	Jaffe (3)	Mahalanobis (4)
Bubble x Spill-SIC	0.152***	0.200***	0.004	−0.000
	(0.027)	(0.037)	(0.009)	(0.013)
Spill-SIC	−0.088***	−0.103***	−0.021***	−0.021**
	(0.016)	(0.033)	(0.006)	(0.010)
Spill-Tech	0.405***	0.844***	0.175***	0.159***
	(0.145)	(0.174)	(0.025)	(0.040)
Fixed Effects	Y, F	Y, F	Y, F	Y, F
Observations	8,896	8,946	8,775	8,825
R^2	0.74	0.74	0.99	0.99

Note: Table 5 presents panel regressions of firm value (Tobin's q or log of sales) on a measure of competition from Bloom, Schankerman, and Van Reenen (2013) interacted with a dummy from Greenwood, Shleifer, and You (2018) that captures whether the firm is in an industry that is in a bubble or not. We control for the technological spillover measure that corresponds to public firms that issue patents in similar technological space. We follow the specification from Tables III and V of Bloom, Schankerman, and Van Reenen (2013) for the Tobin's q and sales regressions, respectively. We also include fixed effects for firm F and patent grant year Y. Standard errors clustered at the year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

ing these additional predictions would require a more sophisticated econometric analysis, which is beyond the scope of this paper.

5 How Other Spillovers Interact with Speculation

Competition is only one of the many ways through which the effect of an innovation are felt in the economy. When a firm innovates, it not only affects shareholders but also workers (see, e.g., Kogan et al. (2020) for recent evidence). Moreover, a new product alters consumers' choice over their entire consumption basket (Blanchard and Kiyotaki (1987)). Finally, others can learn from this innovation, as emphasized by Romer (1986). We incorporate these three prominent sources of spillovers in our theory.

Doing so is interesting for three reasons. First, we obtain new testable predictions of our theory on the interaction of speculation and innovation that can help discipline future empirical work. Second, by comparing across many different specifications, we obtain a more systematic typology of what drives the divergence between market-based and outcome-based measures arising from speculation. Finally, the results highlight the flexibility of our theoretical framework, which is reinforced in Appendix D through many more realistic extensions.[26] Derivations and proofs are presented in Appendix B.

5.1 Model with Richer Spillovers

In all the settings that follow, we maintain the same preferences, beliefs, firm-creation technology, and allocation of production slots as in the model of Section 2. However, we now endogenize how profits $\pi(a)$ are determined in equilibrium, which creates new sources of spillovers. These spillovers, while important, do not change qualitatively the aggregate behavior of the economy in response to speculation. Specifically, an increase in disagreement n always yields a bubble and increases the market-based private value of innovation (in absolute terms and relatively to the outcome-based measure). For this reason, we focus only on the novel predictions for spillovers below.

5.1.1 Labor

Workers supply the labor that is necessary to operate the firms and thus take advantage of blueprints. We formalize this in the model: households are endowed with a fixed quantity of labor L. Firms use labor to produce a homogeneous good according to a technology with decreasing returns to scale.[27] The production function for a given productivity level a is:

$$y(a) = a \cdot \frac{\sigma}{\sigma - 1} \ell^{\frac{\sigma-1}{\sigma}}, \tag{21}$$

where y is firm output, ℓ is firm labor input, and the parameter $\sigma \in [1, \infty]$ controls the returns to scale in labor. In equilibrium (see Appendix B.1), labor trades at a competitive wage w, and the profit function of a firm with productivity a is:

$$\pi(a) = \frac{1}{\sigma - 1} \cdot w^{1-\sigma} a^{\sigma}. \tag{22}$$

As in Section 2, profits are isoelastic with respect to productivity a.

The presence of workers gives rise to two new sources of spillovers. First, workers capture some of the surplus from better innovations through higher

[26] In particular, we study the role of an elastic supply of labor input, a variable number of producing firms M, a setup in which firms compete to participate, and a setting where fixed costs determine the set of producing firms, as in Melitz (2003). For all these models, we obtain simple generalizations of the spillover formula of Proposition 3 and show that the comparative statics of the spillovers remain valid.

[27] This perfectly competitive approach, introduced in Hellwig and Irmen (2001), differs from the most commonly used models with imperfect substitution and monopolistic competition. We study these models and the role of the demand complementarities they induce in Section 5.1.2.

wages. Investors on financial markets do not take into account this value when deciding on new firm creation. This externality is commonly known as the *appropriability effect*. Second, the competition of firms for the same source of labor leads to an interaction between them beyond business-stealing. When a new firm enters, it pushes wages up. This *input price effect* makes labor more expensive for every other firm. For example, the entry of new technology firms such as Facebook creates new opportunities for software engineers (the appropriability effect), bidding their wages up and making it more difficult for other firms to hire (the input price effect). More generally, these spillovers also apply to other production inputs: ℓ could stand for intermediate goods in scarce supply.

Measuring the spillovers. We extend our measure of social value to account for the presence of labor. For the market-based social value of an extra firm, we compute the change in expected utility of households created by this new firm. This calculation includes both the market value of profits from investing in the firms and the income from working in these firms. In our model, it is a dollar amount because of the presence of a quasi-linear good at date 0. The outcome-based social value of an extra firm is still the change in total output in the economy. Importantly, both definitions coincide with the work in Section 3 in the absence of labor.

This overall measure sums the effect of each type of spillover. Here the sources are the appropriability, input price, and business-stealing effects. Each additional source of spillovers introduces extra terms. The separation of different types of spillovers in the overall measure maps naturally to estimation strategies from the empirical literature. For example, as described in Section 4.3, Bloom, Schankerman, and Van Reenen (2013) separate the business-stealing and knowledge spillovers. One could also imagine measuring the effect of firm entry on workers, the appropriability effect. Similarly, the response of firm value to innovation by other firms that compete for the same workers but operate in different product markets would identify the input price effect.[28]

Predictions. The business-stealing effect of the model in Section 2 is still present in this economy given the similar market structure. As before, this spillover, which arises because entrants displace marginal firms, has magnitude $\mathcal{E}_{\mathcal{I}_n}$. Hence, the predictions for this spillover we made in Section 3.2 still hold.

However, we now also have an appropriability effect. When new firms enter, aggregate output increases, with constant elasticity $\mathcal{E}_{\mathcal{C}} = 1/\gamma$. The production function implies that workers receive a constant fraction of aggregate output given by the labor share $(\sigma-1)/\sigma$. Therefore, the social value of entry for workers is $\frac{\sigma-1}{\sigma}\mathcal{E}_{\mathcal{C}}\mathcal{C}/M_e$. Importantly, this quantity does not depend directly on the amount

[28]For clarity, we do not include multiple sectors in the model. The additive property of total spillover with respect to each type of spillover allows us to isolate the different effects.

of disagreement: all firms offer the same wage in equilibrium, so beliefs are irrelevant for workers. However, the market-based private value of the firm does. This value is $\mathcal{I}_n/\mathcal{I}_1 \times \sigma^{-1}\mathcal{C}/M$: the relative expected output of a favorite firm to an average firm multiplied by average profits of firms. The spillover is the ratio of these two values,

$$\text{Appropriability Spillover} = (\sigma - 1)\ \mathcal{E}_{\mathcal{C}}\ \mathcal{I}_1/\mathcal{I}_n. \tag{23}$$

When focusing on the outcome-based measure, we can just replace n with 1, and the spillover becomes $(\sigma - 1)\ \mathcal{E}_{\mathcal{C}}$. According to either metric, the appropriability effect is a positive spillover, larger when workers capture more of total surplus—high σ—or when entry has a stronger impact on output—high $\mathcal{E}_{\mathcal{C}}$. Novel to our model, we see that disagreement does not affect the outcome-based spillover. In contrast, the market-based spillover decreases in n, disappearing altogether in the limit when $n \to \infty$. This is because the bubble inflates the market value of firms, but workers' acknowledge that their earnings are determined by the average producing firm and realize that not all firms will be winners.

Finally, the input price spillover comes from the effect of firm entry on wages. From equation (22), we see that profits have an elasticity of $1 - \sigma$ relative to the wage. In addition, the wage grows as fast as aggregate output, $\mathcal{E}_{\mathcal{C}} = \mathcal{E}_w$, because of the constant labor share and labor supply. A change in wage affects all firms proportionally, irrespective of their productivity. Hence, disagreement does not impact the input price spillover. Rather, both market-based and outcome-based measures of the spillovers are constant and equal to:

$$\text{Input Price Spillover} = -(\sigma - 1)\mathcal{E}_{\mathcal{C}}. \tag{24}$$

This negative externality is larger when firms rely more on labor—high σ—or when the economy responds more to entry—high $\mathcal{E}_{\mathcal{C}}$.[29]

5.1.2 Aggregate Demand

With goods that are not perfect substitutes, households prefer a consumption basket that is diversified. Higher productivity for any particular good thus increases aggregate demand. This implies not only more profit for the specific firm but also for other firms producing the rest of the consumption basket. To formalize the role of aggregate demand, we study an economy with differentiated goods, where firms operate under monopolistic competition at date 1 in the style of Dixit and Stiglitz (1977). Each firm produces a differentiated variety

[29] In the case of agreement, $n = 1$, the appropriability and input price spillovers exactly cancel out. It is a situation where pecuniary externalities cancel out even though the first welfare theorem does not hold because of business-stealing.

indexed by i, and household utility over the set of goods produced is:

$$\mathcal{C} = \left(\int_0^{M_e} \int_{F^{-1}\left(1-\frac{M}{M_e}\right)}^{\infty} c\left(a,i\right)^{\frac{\sigma-1}{\sigma}} dF(a)di \right)^{\frac{\sigma}{\sigma-1}}. \tag{25}$$

Firms operate a linear technology in labor, and output for a firm with productivity a is $y = a\ell$. We leave our other assumptions unaltered.

At the aggregate level, the economy behaves similarly to the previous model.[30] However, the microeconomics of firms' interactions is different and so are profits:

$$\pi(a) = \frac{1}{\sigma} \cdot \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} w^{1-\sigma} \cdot \mathcal{C} \cdot a^{\sigma-1}. \tag{26}$$

The elasticity of profits to individual firm productivity is no longer σ but rather $\sigma - 1$. However, profits are now increasing in aggregate demand $\mathcal{C}$ because of imperfect substitution across goods.

This role of aggregate demand gives rise to an additional source of spillovers: a demand externality as in Blanchard and Kiyotaki (1987). Similar to the role of the wage, aggregate demand affects all firms proportionally. Therefore the demand spillover is the product of the elasticity of aggregate output to entry $\mathcal{C}$ and the elasticity of profits to aggregate output:

$$\text{Demand Spillover} = \mathcal{E}_{\mathcal{C}}. \tag{27}$$

Demand spillovers do not depend on disagreement and are identical whether measured using outcomes or market value.

5.1.3 Knowledge Spillovers

Firms also learn from each other's innovations. We capture the role of knowledge spillovers in the style of Romer (1990) by assuming that a firm's productivity combines its own type, a, and an aggregate of all the active firms' productivity, A. We assume the aggregator is homogeneous of degree one in the productivity distribution.[31] The production function extends (21) and becomes:

$$y = \frac{\sigma}{\sigma-1} a^{1-\alpha} A^{\alpha} \ell^{\frac{\sigma-1}{\sigma}}, \tag{28}$$

[30] The profit share is $1/\sigma$, the aggregate production function is homogeneous of degree one in the distribution of productivities, and the relative labor allocations are efficient. This result was first shown in Lerner (1934). It is the consequence of the homogeneous distortions at the firm level when markups are constant. The macroeconomic elasticities of aggregate consumption and wages to firm entry are therefore $\mathcal{E}_{\mathcal{C}} = \mathcal{E}_w = 1/\gamma$.

[31] In Appendix B.3, we derive the case of Hölder mean of degree q, $A = \left(M_e/M \int_{\underline{a}}^{\infty} a^q dF(a)\right)^{1/q}$, where the parameter $q < \gamma$ controls how much aggregate knowledge comes from the top firms.

where α is the intensity of knowledge spillovers. Again, the macroeconomic features of the simple economy with labor are preserved.[32] However, the microeconomics of firms' interactions differ. Profits are:

$$\pi(a) = \frac{1}{\sigma - 1} \cdot w^{1-\sigma} \cdot A^{\alpha\sigma} \cdot a^{(1-\alpha)\sigma}. \tag{29}$$

As the role of knowledge increases—larger α—firm profits respond more to aggregate knowledge A instead of individual productivity a.

The role of aggregate knowledge for profits highlights how knowledge spillovers operate. Just like the previous two cases, the impact of knowledge on profits is the same irrespective of each individual firm's productivity. The knowledge spillover is therefore the product of the elasticity of profit to knowledge $\alpha\sigma$, and the elasticity of knowledge to entry $1/\gamma$[33]

$$\text{Knowledge Spillover} = \alpha(\sigma - 1)\mathcal{E}_{\mathcal{C}}. \tag{30}$$

This expression does not depend on disagreement and is identical for market-based and outcome-based measures.

5.2 Social Value and Disagreement

The previous results provide useful predictions for how each channel through which innovation affects the economy varies with disagreement. We now turn to the behavior of social value overall, i.e., the total effect of an innovation, which combines all sources of spillover. This quantity is interesting in its own right, but also because it represents the optimal subsidy or tax on firm entry. The following proposition shows that the spillovers we have introduced so far can be organized in three categories, each with its own response to disagreement.

Proposition 3. *For all the models of Section 5, the market-based spillover is:*

$$spill_{mkt}(n) = \underbrace{\mathcal{E}_{\mathcal{I}_n}}_{\textit{bus. stealing}} + \underbrace{\mathcal{E}_{\pi}}_{\textit{general equilibrium}} + \underbrace{(\sigma - 1)\mathcal{E}_{\mathcal{C}} \frac{\mathcal{I}_1}{\mathcal{I}_n}}_{\textit{appropriability effect}}. \tag{31}$$

The outcome-based spillover, which does not depend on disagreement, is:

$$spill_{out} = \mathcal{E}_{\mathcal{I}_1} + \mathcal{E}_{\pi} + (\sigma - 1)\mathcal{E}_{\mathcal{C}}. \tag{32}$$

This decomposition highlights how disagreement can matter for market-based measures of spillovers. While our theoretical exercise is not exhaustive,

[32]The labor share is $(\sigma - 1)/\sigma$ and $\mathcal{E}_{\mathcal{C}} = \mathcal{E}_w = 1/\gamma$.

[33]Because both knowledge and output are determined by the distribution of firm productivity, they grow at the same pace with entry $\mathcal{E}_A = \mathcal{E}_{\mathcal{C}}$.

most spillovers considered in the innovation literature fit in one of our three categories.

The first category is business-stealing. As we have discussed before, disagreement dampens the effect of business-stealing. The risk of displacement is particularly acute for relatively less productive firms. However, with speculation, each investor places a relatively higher weight on her investments having high productivity. In the limit when n goes to infinity, this force can be so strong that the spillover disappears altogether.[34] Competitive interactions between firms of different productivities can take different forms than the displacement of our model. Regardless, the takeaway is that spillovers that are more bottom-heavy tend to be dissipated by disagreement.

This stands in contrast to the second category: general equilibrium effects. In our models, these are the effects of the wage, aggregate demand, and aggregate knowledge on firm profits. The common force across all these sources of spillovers is that they affect all firms proportionally, irrespective of their productivity. Therefore, they can be summarized by the elasticity of firm profits to firm entry, holding productivity constant, $\mathcal{E}_{\pi}$. Because the response to these general equilibrium forces does not interact with the productivity distribution, these spillovers do not depend on beliefs.

Finally, the third category of spillovers is appropriability effects. Not all spillovers affect firms. In our models, workers capture some of the surplus due to firm entry. Because the surplus of workers is determined in the spot market for labor, it does not depend on the relative positions of firms. Unlike firm valuations, wage expectations are not affected by speculation about the relative positions of firms beyond its direct impact on entry and overall labor demand. When disagreement increases, the market-based spillover to workers disappears. This insight is not specific to workers but rather affects all stakeholders of the innovation process. Other key stakeholders, which we could have similarly introduced in the model, include owners of production inputs in scarce supply other than labor or consumers who enjoy some of the surplus.

5.3 Three Illustrations of the Role of Disagreement

We draw three implications from Proposition 3 that illustrate that disagreement fundamentally alters how to measure and interpret the value of innovation.

5.3.1 Macroeconomic versus Microeconomic Elasticities

Without disagreement, market-based and outcome-based measures of spillovers coincide because valuations are expected outcomes. In the economies we have considered, the result is even stronger. The spillover under agreement is the

[34] In the model of Section 2, this occured when $\gamma\theta > \eta$, while now the condition is $\gamma\theta > 1$.

same across all specifications: with labor only, with aggregate demand, and with aggregate knowledge. It takes the value:

$$spill_{out} = spill_{mkt}(1) = \sigma\mathcal{E}_{\mathcal{C}} - 1. \tag{33}$$

While we can derive this expression from equation (32) separately for each model, a simple macroeconomic argument justifies the result. The total effect of entry is the response of aggregate output to entry $d\mathcal{C}/dM_e$. Ex ante, each firm contributes an equal fraction to total output, and the value of a firm is output times the profit share $1/\sigma$. Hence, the value of a firm is $\mathcal{C}/(M_e\sigma)$, and the spillover is given immediately by equation (33).

Regardless of the nature of firm interactions, only two macroeconomic quantities are needed to evaluate the total spillover—the capital share and the elasticity of aggregate output to firm entry. In particular, all our specifications lead to the same values of these two quantities. Disagreement breaks this result. Because different spillovers respond differently to disagreement, the aggregate reasoning under agreement no longer works.

One particularly telling example is the limit of large n when the market-based spillover converges to the profit elasticity $\mathcal{E}_\pi$. This spillover measure is a microeconomic elasticity. It is the response of the profits of one specific firm (i.e., of given productivity) to overall entry. This implies that one needs firm-level data rather than aggregate data to estimate spillovers. Moreover, across our three model specifications, while the outcome-based spillover is identical, the market-based spillover for large disagreement takes different values:

$$spill_{mkt}(n \to \infty) = -\frac{\sigma - 1}{\gamma} \qquad \text{with labor only,} \tag{34}$$

$$spill_{mkt}(n \to \infty) = -\frac{\sigma - 2}{\gamma} \qquad \text{with aggregate demand,} \tag{35}$$

$$spill_{mkt}(n \to \infty) = -\frac{(1-\alpha)\sigma - 1}{\gamma} \qquad \text{with aggregate knowledge.} \tag{36}$$

In other words, the nature of microeconomic interactions matters for market-based spillovers in the presence of disagreement.

5.3.2 Reversal of Comparative Statics

The divergence between market-based and outcome-based spillovers is not only quantitative but also qualitative. Key properties of the economy often have an opposite impact on the total spillover, depending on whether it is measured using outcomes or market values. The following proposition highlights one such reversal, for a parameter common to all of our specification, σ.

Proposition 4. *For all models, the outcome-based spillover is increasing in the labor share. Conversely, with high disagreement ($n \to \infty$ and $\theta > 1/\gamma$), the market-based spillover is decreasing in the labor share.*

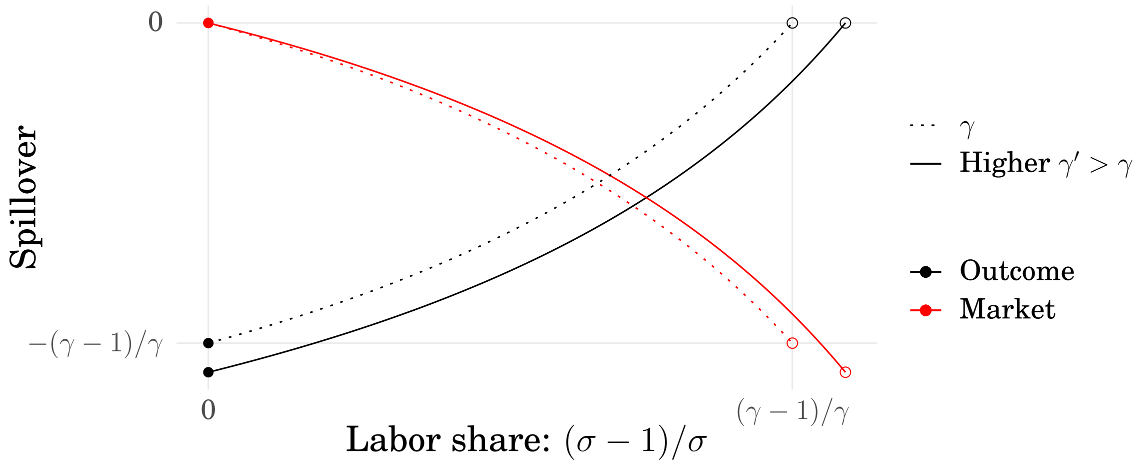

Figure 4
Market-based and outcome-based spillovers.
The figure reports the outcome-based (black lines) and market-based spillovers (red lines) as a function of the labor share for the model with labor only. Solid lines correspond to a larger value of γ than dotted lines.

The outcome-based spillover is given by the macroeconomic formulation of equation (33). An economy with a larger labor share has mechanically a lower capital share. Thus, the importance of social value relative to the value of one firm is larger. For the market-based spillover, the focus is on the elasticity of individual firm profits to entry. A higher labor share implies higher reliance on labor and therefore stronger negative spillovers through the wage effect.

By examining our results, the reader can find more situations where reversals occur. In the model with labor, comparative statics with respect to the thickness of the tail of the productivity distribution γ are also reversed. Figure 4 illustrates these results. Appendix D has more examples.

5.3.3 Reversal of Sign of the Spillover

More strikingly, we also identify situations where the sign of the total spillover is reversed. These are cases where firm entry brings positive externalities according to market-based measures but negative externalities according to outcome-based measures or vice-versa.

Proposition 5. *With demand externalities or knowledge spillovers, if the labor share is close to zero, the outcome-based spillover is positive and the market-based spillover is negative with large disagreement. The converse happens when the labor share is close to its upper bound.*

When the labor share is low, the labor surplus is relatively small, and the dominant force for the wedge is that firms do not internalize the aggregate decreasing returns to scale of the economy, leading to negative real spillovers. With disagreement however, since firms do not rely much on labor, the general equilibrium effect is small. Hence, the demand or knowledge externality dominates, leading to positive value spillovers. The sign reversal across measures of spillovers is not a knife-edge case. Reversals happen throughout the entire range of the labor share whenever $\gamma = 2$ with demand externalities or $\alpha = 1 - 1/\gamma$ with knowledge spillovers. The proposition also shows that the sign reversal can happen in both directions: a positive spillover becoming negative or a negative spillover becoming positive.

6 Welfare Implications

One of the reasons why measuring innovative spillovers is important is because such estimates can guide the design of policies. In this section, we study the welfare implications of spillover measures. We focus on a simple question: If a planner can choose a subsidy or tax on firm entry, what would it be? The presence of disagreement complicates answering this question because the choice of planner objective is not trivial. Which beliefs should the planner use when evaluating allocations?

We follow two approaches: a Pareto criterion and a paternalistic approach imposing common beliefs. For each approach we map the market-based and outcome-based measures of the private and social values we have studied so far to an optimal tax rate. These results further emphasize the importance of acknowledging the presence of speculation and understanding the divergence between measurement approaches.

6.1 Non-Paternalistic Approach

A classic approach to welfare evaluation is the Pareto criterion. An allocation improves overall welfare if all agents in the economy favor it. This criterion is not paternalistic: moving away from the competitive equilibrium requires the support of all agents in the economy. In other words, the optimal tax policy under the Pareto criterion is one that would receive support by all agents in a vote.[35] Interestingly in our case, this approach does not take a stand on what are correct beliefs but rather lets each individual agent evaluate their own utility.

Not having to choose beliefs is a useful feature in the situations we study. We are interested in episodes when there is little information about new firms. Households thus rely on their priors to evaluate these firms, and there is no

[35] This interpretation provides a positive—as opposed to normative—view of these results as the outcome of political decision-making.

way to forecast who is correct. The Pareto criterion respects this difficulty. The planner is no better judge of firms' futures than any investor. In contrast, a paternalistic approach might be more appropriate when heterogeneous beliefs are viewed as inherently inefficient due to a failure of communication or the irrationality of some agent.

Our model is symmetric: despite agents having different beliefs, they each have the same evaluation of their own utility. This symmetry facilitates solving for the optimal allocations because the problem reduces to maximizing the utility of all agents in the economy. The planner chooses a tax τ for each additional firm created from blueprints, rebated as a lump sum to households.[36] The optimal tax can be obtained from a standard Pigouvian taxation problem:

$$\max_{\tau} \mathcal{U}_j = 1 + M_e \mathbf{E}^j \{\pi_i\} - W(M_e). \tag{37}$$

The planner chooses the tax rate that equalizes private incentives to create firms with their social value according to the planner's objective. When a firm creator introduces a new firm, it pays $(1-\tau)p_i$. Hence, the optimal tax rate is $\tau = 1 - \frac{\text{social value}}{p_i}$. Social value for the non-paternalistic planner is given by changes in expected utility, which correspond to market prices. This yields the following simple expression for the optimal tax:

$$\tau_{pareto} = -spill_{mkt}. \tag{38}$$

Therefore, for a non-paternalistic planner, the market-based spillover, rather than the outcome-based spillover, is a sufficient statistic for the optimal tax.

We have seen that the market-based spillover is often very different with agreement compared with disagreement. Our framework thus implies that one should not use estimates from normal market conditions to draw policy conclusions during bubbles. From a positive perspective, the distinction explains why policies to reduce firm entry may not receive support even during bubbles, even when investors agree that there is excess entry. Further, if the only available data is on outcomes, one should use a framework like ours to convert these estimates into their market-based counterparts. The reversal results of the previous section highlight the importance of having a theoretical framework to do so.

6.2 Paternalistic Approach

Alternatively, one could follow a paternalistic approach by assuming that the planner knows the "true" distribution and evaluates allocations under this dis-

[36] We could also consider a more general constrained planner problem. In this general version, the planner allocates date-1 consumption. Consumption plans must be linear combinations of firm profits with positive coefficients. The positive coefficients reflect our assumptions ruling out short-selling and derivative contracts. In the model in Section 2, the allocation chosen by a planner with a linear tax is Pareto optimal for the general constrained planner. It is the most efficient allocation if we impose additionally that welfare weights are equal across agents.

tribution. Brunnermeier, Simsek, and Xiong (2014) propose an improvement on this approach that avoids taking a stance on the true distribution. In their work, the planner considers efficiency across any convex combination of agents' beliefs. In our setting, all agents agree on the distribution of firm productivity, and therefore, this distribution is the only choice for the paternalistic planner. A benefit of the paternalistic approach is that, from the perspective of social choice, it resolves the tension that not all beliefs can be right at the same time.

The paternalistic planner maximizes expected utility under the population distribution F, which coincides with aggregate consumption net of entry costs, $\mathcal{C} - W(M_e)$. As before, this is a standard Pigouvian taxation problem, except that now social value is the marginal effect of entry on output rather than on households' perception of utility. Hence, the optimal tax becomes:

$$\tau_{pater} = 1 - (spill_{out} + 1)\frac{\text{private value}_{out}}{\text{private value}_{mkt}}. \tag{39}$$

Two aspects drive the choice of policy. First, the planner accounts for spillovers. Here the relevant measure of spillovers uses outcomes rather than market values. Similar to the Pareto planner problem, directly using estimates of spillovers measured using market values leads to incorrect inference. If these estimates are the only ones available, one should use a theoretical framework to convert them in outcome-based spillovers.

Second, there is a wedge between the private value on markets, which shapes incentives to create firms and the private value according to the planner's beliefs. We have seen that the bubble inflates market-based private value relative to outcome-based private value. As a result, this second force will generally push toward taxing entry in order to lean against what the planner views as excessive valuations. In particular, if outcome-based spillovers are positive, implying that there is under-entry absent disagreement, a bubble can actually push the economy toward efficiency.

With agreement, the two planner objectives coincide and lead to the same optimal tax. The optimal tax in this case is the negative of the total spillover, regardless of whether it is measured using market- or outcome-based measures.

7 Conclusion

Speculation and innovation often coincide. A naive view of the role of speculation and the bubbles it generates is that they do not matter for the innovation process even though they disrupt financial markets. In contrast, we argue that there is structure in these disruptions—innovation and speculation interact systematically. Failing to understand and account for this interaction can distort our qualitative and quantitative understanding of the private and social values of innovation, leading to erroneous answers to both positive and normative questions about innovation.

To that end, the paper introduces a model of the interaction between innovation and speculation. Our theory makes sharp predictions for the value of innovation under market- and outcome-based measures. We study the impact of disagreement on both the private and social values of innovation. The structure of the model reflects reality. During bubbles, information from financial markets and real outcomes about the value of innovation diverge from each other in the direction predicted by our theory. Extending the work of Kogan et al. (2017), we find that asset prices indicate increases in the private value of innovation during bubbles, with no commensurate increase in outcome-based measures. Using the method of Bloom, Schankerman, and Van Reenen (2013), we show that competitive spillovers disappear during bubbles when measured in financial markets but remain unchanged when looking at sales.

Thus, accounting for the link between speculation and innovation is a fruitful enterprise. Our paper provides the first steps toward digging deeper, showing how our framework can entertain many sources of innovative spillovers and highlighting broad principles of their interaction with disagreement. We also explain how to map data to innovation policy decisions in periods of disagreement. Another avenue suggested by our work is the use of financial market regulation as a novel form of innovation policy.[37] We look forward to seeing the future implementation of these ideas improve our understanding of the outcomes of innovation.

[37] Jørring et al. (2017) discuss the example of Food and Drug Administration hedges to share risk in medical innovation. Our results show that disagreement can dramatically change the consequences of such instruments both in terms of outcomes and welfare.

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Appendix

A Derivations for Simple Model

A.1 General Results

We first express the spillovers in integral form, then prove Proposition 1 for general functions F and π.

A.1.1 General Formulas

The market-based spillover is

$$spill_{mkt}(n; M_e) = [M_e V^{(n)\prime}(M_e)]/[V^{(n)}(M_e)].$$

The numerator and denominator have interpretable expressions. First rewrite the denominator in the following integral form:

$$\begin{aligned} V^{(n)} &= \int_{F^{-1}}^{\infty} \pi(x) dF_n(x) \\ &= \int_{F^{-1}}^{\infty} \pi(x) \frac{F_n'}{F'}(x)\, dF(x), \end{aligned} \tag{IA.1}$$

where we denote $F^{-1}(1 - M/M_e)$ by F^{-1} for convenience. Now the numerator can be written:

$$M_e \frac{dV^{(n)}}{dM_e} = \frac{M}{M_e} \cdot \pi\left[F^{-1}\right] \cdot \frac{F_n'}{F'}\left[F^{-1}\right] \tag{IA.2}$$

$$= -\int_{F^{-1}}^{\infty} \pi\left[F^{-1}\right] \frac{F_n'}{F'}\left[F^{-1}\right] dF(x). \tag{IA.3}$$

This leads to the following formula for the market-based spillover:

$$spill_{mkt}(n) = -\frac{\int_{F^{-1}}^{\infty} \pi\left[F^{-1}\right] \frac{F_n'}{F'}\left[F^{-1}\right] dF(x)}{\int_{F^{-1}}^{\infty} \pi(x) \frac{F_n'}{F'}(x)\, dF(x)}. \tag{IA.4}$$

A.1.2 Comparing Spillovers

Lemma A.1. *Holding M_e constant, the market-based spillover is larger with agreement than with disagreement.*

Proof. First recall the market-based spillover $spill_{mkt}(n; M_e) = \frac{M_e V^{(n)\prime}(M_e)}{V^{(n)}(M_e)}$. We have the derivative:

$$V^{(n)\prime} = -\frac{1}{M_e}\frac{M}{M_e}\pi\left[F^{-1}\right] \cdot n\left(1 - \frac{M}{M_e}\right)^{n-1},$$

and we can bound $V^{(n)}$:

$$\begin{aligned} V^{(n)} &= \int_{F^{-1}}^{\infty} \pi(x) n F^{n-1}(x) dF(x) \\ &\geq \int_{F^{-1}}^{\infty} \pi(x) n F^{n-1}\left[F^{-1}\right] dF(x) \\ &\geq n\left(1-\frac{M}{M_e}\right)^{n-1} \int_{F^{-1}}^{\infty} \pi(x) dF(x). \end{aligned}$$

Therefore, we are able to bound the market-based spillover for a given M_e and n:

$$|spill_{mkt}(n; M_e)| \leq -\frac{\int_{F^{-1}}^{\infty} \pi\left[F^{-1}\right] dF(x)}{\int_{F^{-1}}^{\infty} \pi(x) dF(x)} \leq |spill_{mkt}(1; M_e)|, \tag{IA.5}$$

where the second inequality comes from the definition of $spill_{mkt}(1; M_e)$. ∎

A.2 Power Case Derivations

We now outline the derivations for the case we focus on in the main text, with $F(a) = 1 - a^{-\gamma}$ and $\pi(a) = a^{\eta} \cdot \mathbf{1}\{a \geq \underline{a}\}$.

First, define

$$\underline{a} = F^{-1}(1 - \frac{M_e}{M}) = \left(\frac{M_e}{M}\right)^{1/\gamma}.$$

The ex-ante value of a firm $V^{(n)}(M_e)$ is:

$$\begin{aligned} V^{(n)}(M_e) &= \int_{\left(\frac{M_e}{M}\right)^{\frac{1}{\gamma}}}^{\infty} x^{\eta} \gamma n x^{-\gamma-1}\left(1-x^{-\gamma}\right)^{n-1} dx && \text{(IA.6)} \\ &= \gamma n \underline{a}^{\eta-\gamma} \int_{1}^{\infty} t^{\eta-\gamma-1}\left(1-\underline{a}^{-\gamma} t^{-\gamma}\right)^{n-1} dt. && \text{(IA.7)} \end{aligned}$$

The first derivative with respect to entrants is:

$$\frac{dV^{(n)}}{dM_e} = -\frac{1}{M_e} \cdot \frac{M}{M_e}\left(\frac{M_e}{M}\right)^{\frac{\eta}{\gamma}} \cdot n\left(1-\frac{M}{M_e}\right)^{n-1}. \tag{IA.8}$$

It is convenient to express $-M_e V^{(n)\prime}(M_e)$ as:

$$-M_e V^{(n)\prime}(M_e) = \underline{a}^{\eta-\gamma} \cdot n\left(1-\underline{a}^{-\gamma}\right)^{n-1}. \tag{IA.9}$$

A.2.1 Spillovers and Firm Entry under Agreement

Lemma A.2. *The outcome-based spillover does not depend on the level of entry:*

$$spill_{out}(M_e) = -\frac{\gamma-\eta}{\gamma}. \tag{IA.10}$$

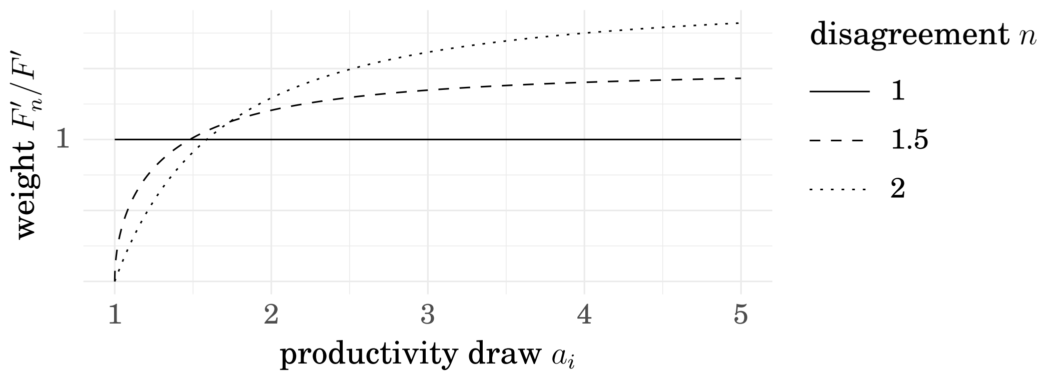

Figure IA.1
Distortion of productivity weights F'_n/F'

The level of entry is:

$$\frac{M_e}{M} = \left(f_e \frac{\gamma - \eta}{\gamma}\right)^{-\frac{\gamma}{\gamma(\theta+1)-\eta}}. \tag{IA.11}$$

Proof. The value of a firm is:

$$\begin{aligned} V^{(1)}(M_e) &= \gamma \underline{a}^{\eta-\gamma} \int_1^\infty t^{\eta-\gamma-1} dt \\ &= \frac{\gamma}{\gamma-\eta} \underline{a}^{\eta-\gamma} = \frac{\gamma}{\gamma-\eta} \left(\frac{M_e}{M}\right)^{\frac{\eta-\gamma}{\gamma}}. \end{aligned} \tag{IA.12}$$

From equation (IA.9) with $n = 1$, we have the numerator of the spillover:

$$-M_e \frac{dV^{(1)}}{dM_e} = \left(\frac{M_e}{M}\right)^{\frac{\eta-\gamma}{\gamma}}, \tag{IA.13}$$

which leads directly to the desired formula (IA.10) for the outcome-based spillover. Finally, we can rewrite equation (8):

$$f_e \left(\frac{M_e}{M}\right)^{\theta} = \frac{\gamma}{\gamma-\eta} \left(\frac{M_e}{M}\right)^{\frac{\eta-\gamma}{\gamma}},$$

which reduces to (IA.11) as desired. ∎

A.2.2 Disagreement Lowers Market-Based Spillovers

Proposition 1 follows directly from Lemma A.1 and Lemma A.2. In particular, we have

$$|spill_{mkt}(n; M_e)| \leq |spill_{mkt}(1; M_e)| = |spill_{out}(M_e)| = |spill_{out}(M'_e)| \tag{IA.14}$$

for any M'_e, and, in particular, the M'_e corresponding to the equilibrium entry under agreement. The inequality is from Lemma A.1; the first equality is from the definition of $spill_{mkt}$ and $spill_{out}$; and the last equality is from Lemma A.2.

A.2.3 Disagreement Asymptotics

Lemma A.3. *If $\theta \geq 0$, then as disagreement increases ($n \to \infty$), the mass of entrants also increases and goes to infinity:* $\lim_{n\to\infty} M_e = \infty$.

Proof. We define $\underline{a}_n = (M_e/M)^{1/\gamma}$, where M_e now depends on n, and show that $\underline{a}_n \to \infty$. Equation (8) implies an implicit definition of the sequence $\underline{a}_n$:

$$f_e \underline{a}_n^{\gamma(\theta+1)-\eta} = \gamma n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}_n^{-\gamma} t^{-\gamma}\right)^{n-1} dt.$$

Suppose $\underline{a}_n$ has a finite limit that is strictly larger than zero, i.e., $\underline{a}_\infty > 0$.[38] Then there exists N large enough such that $\forall n > N$, $\underline{a}_n > A = \underline{a}_\infty - \epsilon > 0$. We obtain a lower bound for the right-hand side of the implicit equation above:

$$\begin{aligned} I_n &= \gamma n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}^{-\gamma} t^{-\gamma}\right)^{n-1} dt \\ &> \gamma n \int_1^\infty t^{\eta-\gamma-1} \left(1 - A^{-\gamma} t^{-\gamma}\right)^{n-1} dt. \end{aligned}$$

Consider an arbitrary threshold T_n that depends on n and satisfies:

$$\begin{aligned} I_n &> \gamma n \int_{T_n}^\infty t^{\eta-\gamma-1} \left(1 - A^{-\gamma} t^{-\gamma}\right)^{n-1} dt \\ &> \gamma n \left(1 - A^{-\gamma} T_n^{-\gamma}\right)^{n-1} \int_{T_n}^\infty t^{\eta-\gamma-1} dt \\ &= \frac{\gamma}{\gamma-\eta} \cdot n \cdot T_n^{\eta-\gamma} \left(1 - A^{-\gamma} T_n^{-\gamma}\right)^{n-1}. \end{aligned}$$

Choose the threshold $T_n = n^{1/\gamma}$. The bound becomes:

$$\begin{aligned} I_n &> \frac{\gamma}{\gamma-\eta} \cdot n^{\frac{\eta}{\gamma}} \exp\left(-(n-1)\log(1 - A^{-\gamma} n^{-1})\right) \\ &> \frac{\gamma}{\gamma-\eta} \cdot n^{\frac{\eta}{\gamma}} \exp\left(-(n-1) A^{-\gamma} n^{-1} + \mathcal{O}(n^{-1})\right). \end{aligned}$$

Since $\gamma(\theta+1) - \eta \geq \gamma - \eta > 0$, this implies $I_n \to \infty$, contradicting $\underline{a}_\infty < \infty$. ∎

Lemma A.4 (Asymptotics for firm creation). *In the high-disagreement limit ($n \to \infty$), we have the following asymptotics for the mass of firms created, M_e:*

- *If $\gamma\theta < \eta$, then $M_e/M = \left(\frac{1}{f_e} \frac{\gamma}{\gamma-\eta} \cdot n\right)^{\frac{\gamma}{\gamma(\theta+1)-\eta}}$.*
- *If $\gamma\theta = \eta$, then $\lim_{n\to\infty} M_e/M = \alpha_\infty n$, where α_∞ is a constant defined below.*

Proof. Substituting $\underline{a}$ into (8), we have:

$$\begin{aligned} f_e &= \gamma \underline{a}^{\eta-\gamma(\theta+1)} n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}^{-\gamma} t^{-\gamma}\right)^{n-1} dt \\ &\simeq \gamma \underline{a}^{\eta-\gamma(\theta+1)} n \int_1^\infty t^{\eta-\gamma-1} \exp\left(-(n-1)\underline{a}^{-\gamma} t^{-\gamma}\right) dt, \end{aligned}$$

[38] Since the mass of firms producing cannot be higher than the mass of firms created, $\underline{a}_n \geq 1$.

where we have used the fact that $\underline{a} \to \infty$ from Lemma A.4, and $\log(1-x) = -x + \mathcal{O}(x^2)$. To find a solution, we guess the asymptotics of $\underline{a}(n)$. We rewrite $\underline{a} = \alpha(n)n^{1/(\gamma(1+\theta)-\eta)}$ and show that $\alpha(n)$ converges to a finite limit α. The above equation becomes:

$$f_e = \gamma\alpha(n)^{\eta-\gamma(\theta+1)} \int_1^\infty t^{\eta-\gamma-1} \exp\left(-\alpha(n)^{-\gamma} \frac{n-1}{n^{\frac{\gamma}{\gamma(1+\theta)-\eta}}} t^{-\gamma}\right) dt.$$

Suppose $\gamma\theta < \eta$. Then the exponential term converges to zero and we have:

$$f_e = \gamma\alpha^{\eta-\gamma(\theta+1)} \int_1^\infty t^{\eta-\gamma-1} = \alpha^{\eta-\gamma(\theta+1)} \frac{\gamma}{\gamma-\eta},$$

such that we have the following asymptotics for firm entry:

$$\frac{M_e}{M} = \left(\frac{1}{f_e}\frac{\gamma}{\gamma-\eta} \cdot n\right)^{\frac{\gamma}{\gamma(\theta+1)-\eta}}. \tag{IA.15}$$

Suppose $\gamma\theta = \eta$. Then $\underline{a}$ is defined by:

$$f_e = \gamma\underline{a}_n^{-\gamma} n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}^{-\gamma}t^{-\gamma}\right)^{n-1} dt.$$

Since $\underline{a} = (M_e/M)^{1/\gamma}$, it is sufficient to guess and verify that $\underline{a}_n = \alpha(n)^{-1/\gamma}n^{1/\gamma}$, and $\alpha(n)$ has a finite limit α_∞ defined by:

$$\begin{aligned} f_e &= \gamma\alpha(n) \int_1^\infty t^{\eta-\gamma-1} \exp\left(-(n-1)(\alpha(n)n^{-1} + \mathcal{O}(\alpha(n)^2 n^{-2}))t^{-\gamma}\right) dt \\ &\xrightarrow[n\to\infty]{} \gamma\alpha_\infty \int_1^\infty t^{\eta-\gamma-1} e^{-\alpha_\infty t^{-\gamma}} dt, \end{aligned}$$

where we take the limit when $n \to \infty$. The outcome-based spillover implies:

$$\begin{aligned} f_e &> \gamma\alpha_\infty e^{-\alpha_\infty} \int_1^\infty t^{\eta-\gamma-1} dt \\ &> \alpha_\infty e^{-\alpha_\infty} \frac{\gamma}{\gamma-\eta}, \end{aligned}$$

and thus

$$\frac{\alpha_\infty e^{-\alpha_\infty}}{f_e} < \frac{\gamma-\eta}{\gamma}, \tag{IA.16}$$

which implies a finite bound on α_∞. ∎

Using the asymptotics derived in Lemma A.4, we now prove Proposition 2.

Proof. **(Proposition 2)** Suppose $\gamma\theta < \eta$. Substitute the asymptotics derived in equation

(IA.15) into the formula for the market-based spillover:

$$spill_{mkt}(n; M_e) = -\frac{\frac{M}{M_e}\left(\frac{M_e}{M}\right)^{\frac{\eta}{\gamma}} n\left(1-\frac{M}{M_e}\right)^{n-1}}{f_e\left(\frac{M_e}{M}\right)^{\theta}} \tag{IA.17}$$

$$\simeq -\frac{1}{f_e}\cdot f_e\frac{\gamma-\eta}{\gamma}\frac{1}{n}\cdot n\left(1-\frac{M}{M_e}\right)^{n-1} \to -\frac{\gamma-\eta}{\gamma}, \tag{IA.18}$$

where we have used the fact that $(1-M/M_e)^{n-1}\to 1$.[39] The market-based spillover therefore converges to the outcome-based spillover in this case.

Now suppose $\gamma\theta > \eta$. We write the market-based spillover directly:

$$spill_{mkt}(n; M_e) = -\frac{n\underline{a}^{\eta-\gamma}(1-\underline{a}^{-\gamma})^{n-1}}{f_e\underline{a}^{\gamma\theta}}.$$

First suppose $\underline{a}\to\infty$. We rewrite the competitive equilibrium condition (8):

$$n\underline{a}^{-\gamma} = \frac{f_e\underline{a}^{\gamma\theta-\eta}}{\gamma\int_1^\infty t^{\eta-\gamma-1}(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1}dt}.$$

The denominator is bounded from above by $\gamma\int_1^\infty t^{\eta-\gamma-1}dt$, which implies $n\underline{a}\to\infty$. Using a first-order approximation, we have:

$$(1-\underline{a}^{-\gamma})^{n-1}\simeq\exp\left(-n\underline{a}^{-\gamma}\right).$$

Therefore, the market-based spillover in the limit is:

$$spill_{mkt}(n)\simeq -\frac{n\underline{a}^{-\gamma}\exp\left(-n\underline{a}^{-\gamma}\right)}{f_e\underline{a}^{\gamma\theta-\eta}}\to 0, \tag{IA.19}$$

since the numerator goes to zero and the denominator goes to infinity. Suppose instead that $\underline{a}$ has a finite limit. We obtain the expression for $spill_{mkt}$:

$$spill_{mkt}(n) = -\frac{n(1-\underline{a}^{-\gamma})^{n-1}}{f_e\underline{a}^{\gamma(1+\theta)-\eta}} = -\frac{n\exp\left((n-1)\log(1-\underline{a}^{-\gamma})\right)}{f_e\underline{a}^{\gamma(1+\theta)-\eta}}\to 0, \tag{IA.20}$$

since the denominator has a finite limit and the numerator goes to 0.

Lastly, consider the case where $\gamma\theta=\eta$. The spillover expression simplifies to:

$$spill_{mkt}(n) = -\frac{1}{f_e}\cdot n\underline{a}^{-\gamma}\left(1-\underline{a}^{-\gamma}\right)^{n-1}.$$

[39] This follows from $(1-M/M_e)^{n-1} = \exp\left[-(n-1)\log(M_e/M)\right]$ and using the asymptotics derived above for $\gamma\theta<\eta$: $(1-M/M_e)^{n-1} = \exp\left[-(n-1)\left(f_e^{-1}\frac{\gamma}{\gamma-\eta}n\right)^{-\frac{\gamma}{\gamma(1+\theta)-\eta}}\right]\to 1$.

Using Lemma A.4 and the result that $\underline{a}^{-\gamma} = \alpha(n)/n$, and $\alpha(n) \to \alpha_\infty$, we have:

$$spill_{mkt}(n) \simeq -\frac{1}{f_e}\alpha(n)\exp\left(-(n-1)\alpha(n)/n\right) \tag{IA.21}$$

$$\simeq -\frac{1}{f_e}\alpha(n)\exp\left(-\alpha(n)\right) \to -\frac{1}{f_e}\alpha_\infty e^{-\alpha_\infty}. \tag{IA.22}$$

Moreover, using Lemma A.4, this also proves that in the limit, the magnitude of the market-based spillover $|spill_{mkt}|$ is less than that of the outcome-based spillover $|spill_{out}| = (\gamma - \eta)/\gamma$.
$\blacksquare$

B Derivations for General Equilibrium Model

Recall the definition of the average of a power function in productivity under measure $F^{(n)}$:

$$\mathcal{I}_n(M_e, \sigma) = \int_{\underline{a}}^{\infty} a^\sigma dF^n(a).$$

The integral with no disagreement is $\mathcal{I}_1$. We will use $\mathcal{I}_n$ when the dependence of the integral to M_e or σ is unambiguous. Under the Pareto distribution with parameter γ, we have the following result:

$$\mathcal{I}_1 = \frac{\gamma}{\gamma - \sigma} \cdot \left(\frac{M_e}{M}\right)^{\frac{\sigma}{\gamma}-1}. \tag{IA.23}$$

B.1 Model with Decreasing Returns to Scale

B.1.1 Equilibrium

The firm-optimization problem given the production function and the competitive input price w is:

$$\max_{\ell(a)} \pi(a) = a \cdot \frac{\sigma}{\sigma - 1} \ell(a)^{\frac{\sigma-1}{\sigma}}.$$

The first-order condition leads to demand for labor at the firm level:

$$\ell(a) = \left(\frac{w}{a}\right)^{-\sigma}.$$

Output and profit at the firm level are:

$$\begin{aligned} y(a) &= \frac{\sigma}{\sigma-1} \cdot w^{1-\sigma} \cdot a^\sigma \\ \pi(a) &= \frac{1}{\sigma-1} \cdot w^{1-\sigma} \cdot a^\sigma. \end{aligned}$$

Market clearing on the input market yields:

$$L = M_e \cdot w^{-\sigma} \int_{\underline{a}}^{\infty} a^\sigma dF(a) = M_e \cdot w^{-\sigma} \cdot \mathcal{I}_1, \tag{IA.24}$$

which, given (IA.23), leads to the following wage in equilibrium under a Pareto distribution for F:

$$w = \left(\frac{\gamma}{\gamma-\sigma}\right)^{\frac{1}{\sigma}} \cdot \left(\frac{M_e}{L}\right)^{\frac{1}{\sigma}} \cdot \left(\frac{M_e}{M}\right)^{\frac{1}{\gamma}-\frac{1}{\sigma}}.$$

Given the equilibrium quantities, we can decompose aggregate output into the profit and labor shares. First, observe that aggregate ouput is:

$$\mathcal{C} = M_e \cdot \int_{\underline{a}}^{\infty} y(a) dF(a) = M_e \cdot \frac{\sigma}{\sigma-1} w^{1-\sigma} \mathcal{I}_1.$$

From this expression we immediately conclude that:

$$w^{1-\sigma} \cdot \mathcal{I}_1 = \frac{\sigma - 1}{\sigma} \cdot \frac{\mathcal{C}}{M_e},$$

and we are able to simplify the ex-ante valuation of firms:

$$\begin{aligned} V^{(n)}(M_e) &= \int_{\underline{a}}^{\infty} \pi(a) dF^n(a) = \frac{1}{\sigma - 1} \cdot w^{1-\sigma} \cdot \mathcal{I}_n(M_e) \\ &= \frac{1}{\sigma} \cdot \frac{\mathcal{C}}{M_e} \cdot \frac{\mathcal{I}_n(M_e)}{\mathcal{I}_1(M_e)}. \end{aligned}$$

Finally, we express the wage as a function of the equilibrium mass of firms:

$$w = \left(\frac{\gamma - \sigma}{\gamma} L \right)^{-\frac{1}{\sigma}} \cdot M^{\frac{\gamma - \sigma}{\gamma \sigma}} \cdot M_e^{\frac{1}{\gamma}}.$$

The equilibrium condition that determines entry in equilibrium is:

$$W'(M_e) = \frac{1}{\sigma} \cdot \frac{\mathcal{C}}{M_e} \cdot \frac{\mathcal{I}_n(M_e)}{\mathcal{I}_1(M_e)}. \tag{IA.25}$$

B.1.2 Spillovers

As before, the spillover is a ratio of the direct and indirect effect of firm entry. Consumption for household j is the product of labor income and profits from its investment:

$$\mathcal{C}_j = \underbrace{\frac{\sigma - 1}{\sigma} \cdot \mathcal{C}}_{\text{labor income: } wL} + \underbrace{\frac{1}{\sigma} \cdot \frac{\mathcal{I}_n}{\mathcal{I}_1} \cdot \mathcal{C}}_{\text{firm profits: } V^{(n)}}.$$

Hence the social value under market-based measures of entry can be written:

$$\frac{1}{\sigma} \cdot \frac{d\left(\mathcal{I}_n / \mathcal{I}_1 \cdot \mathcal{C}\right)}{dM_e} + \frac{\sigma - 1}{\sigma} \cdot \frac{d\mathcal{C}}{dM_e}. \tag{IA.26}$$

To find the market-based spillover, we take the ratio of the direct and indirect effect of firm entry:

$$1 - spill_{mkt}(n) = \frac{M_e}{\mathcal{C}} \frac{\mathcal{I}_1}{\mathcal{I}_n} \cdot \frac{d\left(\mathcal{I}_n / \mathcal{I}_1 \cdot \mathcal{C}\right)}{dM_e} + (\sigma - 1) \cdot \frac{M_e}{\mathcal{C}} \frac{\mathcal{I}_1}{\mathcal{I}_n} \cdot \frac{d\mathcal{C}}{dM_e}.$$

This leads us immediately to the general formula for the market-based spillover:

$$spill_{mkt}(n) = -\mathcal{E}_{\mathcal{I}_n} + (1 + \mathcal{E}_{\mathcal{I}_1} - \mathcal{E}_{\mathcal{C}}) - (\sigma - 1)\mathcal{E}_C \cdot \frac{\mathcal{I}_1}{\mathcal{I}_n}. \tag{IA.27}$$

The asymptotic environment is similar to that of Section 2. We start by studying the asymptotics of the business-stealing effect.

Lemma B.1 (Asymptotics for business-stealing effect)**.** *In the high-disagreement limit ($n \to \infty$), the business-stealing effect converges to a limit that depends on the marginal cost of firm creation θ:*

- *If* $\theta\gamma < 1$*, then* $\lim_{n\to\infty} \mathcal{E}_{\mathcal{I}_n} = \mathcal{E}_{\mathcal{I}_1} = \frac{\sigma}{\gamma} - 1$.
- *If* $\theta\gamma > 1$*, then* $\lim_{n\to\infty} \mathcal{E}_{\mathcal{I}_n} = 0$.
- *If* $\theta\gamma = 1$*, then* $\lim_{n\to\infty} \mathcal{E}_{\mathcal{I}_n} = \alpha_\infty e^{-\alpha_\infty}/f_e$.

Proof. The free-entry condition equation (IA.25) leads to:

$$(\sigma - 1)\left(\frac{\gamma - \sigma}{\gamma} L\right)^{\frac{1-\sigma}{\sigma}} \cdot f_e = M^{\theta + \frac{\sigma-\gamma}{\gamma}\frac{\sigma-1}{\sigma}} \cdot M_e^{\frac{1-\sigma}{\gamma} - \theta} \cdot \mathcal{I}_n.$$

We recast the free-entry condition using $\underline{a}$ to be able to use the asymptotic results from Lemma A.4

$$\text{constant} = \underline{a}^{1-\sigma-\gamma\theta} \int_{\underline{a}}^{\infty} x^\sigma dF_n(x).$$

Writing $\tilde{\theta} = \theta + (\sigma - 1)/\gamma$ and $\tilde{\eta} = \sigma$, we recognize the expression from Lemma A.4 and use Proposition 2. ∎

For the labor surplus term, we study the behavior of $\mathcal{I}_1/\mathcal{I}_n$.

Lemma B.2 (Asymptotics for labor surplus distortion)**.** *In the high-disagreement limit ($n \to \infty$), the labor surplus distortion disappears:*

$$\lim_{n\to\infty} (\sigma - 1)\mathcal{E}_{\mathcal{C}} \frac{\mathcal{I}_1}{\mathcal{I}_n} = 0$$

Proof. Since $\tilde{\theta} > 0$, Lemma A.3 gives $\lim_{n\to\infty} M_e = \infty$. The proof of Lemma A.3 implies $\lim_{n\to\infty} \mathcal{I}_n = \infty$. Finally, because $\sigma < \gamma$,

$$\mathcal{I}_1 = \frac{\gamma}{\gamma - \sigma}\left(\frac{M_e}{M}\right)^{\frac{\sigma-\gamma}{\gamma}} \to 0$$

as $n \to \infty$. Therefore, $\mathcal{I}_1/\mathcal{I}_n$ converges to 0. ∎

B.2 Differentiated Goods

B.2.1 Date 1 Economy

The introduction of differentiated goods in Section 5.1.2 changes the production stage. We therefore focus on the equilibrium conditions at date 1.

Firms produce a mass M of differentiated goods, indexed by (a, i), where a is firm productivity and i indexes the firms. We drop the i index when unambiguous. Household utility aggregates consumption of these goods with constant elasticity of substitution σ across goods. At date 1, household j with total expenditure E_j solves:

$$\mathcal{C}(E_j) = \max_{\{c(a,i)\}} \left(\int_0^{M_e} \int_{F^{-1}\left(1-\frac{M}{M_e}\right)}^{\infty} c(a,i)^{\frac{\sigma-1}{\sigma}} dF(a)di\right)^{\frac{\sigma}{\sigma-1}}$$
$$\text{s.t.} \int_0^{M_e} \int_{F^{-1}\left(1-\frac{M}{M_e}\right)}^{\infty} p(a,i)\, c(a,i)\, dF(a)di \le E_j.$$

For reasons that will soon be clear, we denote by $1/\mathcal{P}$ the Lagrange multiplier on the budget constraint. Because the objective function is homogeneous of degree one in consumption and the budget constraint is linear, $\mathcal{C}(E_j)$ is linear in E_j. Thus, we have $\mathcal{C}(E_j) = E_j/\mathcal{P}$. Therefore, $\mathcal{P}$ is the price of one unit of the consumption basket. We use this consumption basket as the numeraire at date 1 by normalizing $\mathcal{P} = 1$. The linearity also implies that to aggregate individual demands, it is sufficient to know the aggregate expenditure in the economy and not the whole distribution of individual expenditures. The first-order condition in the problem above implies the demand curve:

$$c(p) = \mathcal{C}p^{-\sigma}.$$

Output for a firm with productivity a is $y = a\ell$. Firms face monopolistic competition. They maximize profits by setting prices, taking as given the demand curve from each household:

$$\max_{p(a)} \; p(a)y\left(p\left(a\right)\right) - \frac{wy\left(p\left(a\right)\right)}{a} = \mathcal{C}\left[p(a)^{1-\sigma} - \frac{w}{a}p(a)^{-\sigma}\right].$$

The optimal price is therefore

$$p(a) = \frac{\sigma}{\sigma-1}\frac{w}{a}.$$

Firms charge a markup $\sigma/(\sigma-1)$ over their marginal cost w/a.

We can then compute output y, revenue py, labor expenditure $w\ell$, and profits π as functions of productivity:

$$\begin{aligned}
y &= \mathcal{C}w^{-\sigma}a^{\sigma}\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} \\
py &= \mathcal{C}w^{1-\sigma}a^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \\
w\ell &= \frac{\sigma-1}{\sigma}\mathcal{C}w^{1-\sigma}a^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \\
\pi &= \frac{1}{\sigma}\mathcal{C}w^{1-\sigma}a^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}.
\end{aligned}$$

We see that labor expenditure is a fraction $(\sigma-1)/\sigma$ of revenues, and profits make up the remaining $1/\sigma$ share.

Labor market clearing gives $\mathcal{C}(\sigma-1)/\sigma = wL$. In equilibrium, aggregate expenditure is equal to aggregate consumption, so we have:

$$\begin{aligned}
\mathcal{C} &= \mathcal{C}\left(\frac{\sigma}{\sigma-1}w\right)^{1-\sigma} M_e\mathcal{I}_1(M_e, \sigma-1) \\
&= M_e^{\frac{1}{\sigma-1}}\left(\frac{\gamma}{\gamma-(\sigma-1)}\right)^{\frac{1}{\sigma-1}}\left(\frac{M_e}{M}\right)^{\frac{(\sigma-1)-\gamma}{(\sigma-1)\gamma}} \cdot L \\
&= \left(\frac{\gamma}{\gamma-(\sigma-1)}\right)^{\frac{1}{\sigma-1}} M^{\frac{1}{\sigma-1}-\frac{1}{\gamma}} M_e^{\frac{1}{\gamma}} \cdot L.
\end{aligned}$$

Therefore, we have $\mathcal{E}_\mathcal{C} = \mathcal{E}_w = 1/\gamma$. Alternatively, notice that the labor allocation is efficient with monopolistic competition and the aggregate production function is homogeneous of degree one in the distribution of productivity. Because an increase in M_e increases all productivities with an elasticity $1/\gamma$, this results in an elasticity of aggregate consumption of $1/\gamma$.

B.2.2 Spillovers

All arguments behind Proposition 3 apply, so the proposition is still valid, but with $\mathcal{I}_1$ and $\mathcal{I}_n$ now evaluated with parameter $\sigma - 1$.

Because the aggregate consumption elasticity is unchanged, the output-based spillover (and market-based spillover under agreement) is unchanged: $spill_{out} = -(\gamma - \sigma)/\gamma$.

With speculation, the free-entry condition is:

$$\left(\frac{M_e}{M}\right)^{\theta} = \frac{1}{\sigma}\mathcal{C}\left(\frac{\mathcal{C}}{L}\right)^{1-\sigma}\mathcal{I}_n,$$

which we can rewrite as:

$$KM_e^{\theta-(1-(\sigma-1))/\gamma} = \mathcal{I}_n,$$

where K does not depend on M_e and n. This is again the same condition as the homogeneous goods model, with σ replaced by $\sigma - 1$. The condition for the convergence of $\mathcal{E}_{\mathcal{I}_n}$ from Lemma B.1 still applies as well. In the high-disagreement limit with $\theta > 1/\gamma$, the market-based spillover becomes:

$$spill_{mkt}(n \to \infty) = -1 - \mathcal{E}_{\mathcal{I}_1} + \mathcal{E}_{\mathcal{C}} = -\frac{\sigma - 2}{\gamma}. \tag{IA.28}$$

The upper panel of Figure IA.2 shows the sign reversal of the market-based spillover, as described in Proposition 5.

B.3 Knowledge Externalities

B.3.1 Date 1 Economy

We again focus on the date 1 economy and return to a setting with decreasing return to scale.

Now firm productivity depends on the productivity of other firms producing. In particular, consider the production function:

$$y = \frac{\sigma}{\sigma - 1}a^{1-\alpha}A^{\alpha}\ell^{\frac{\sigma-1}{\sigma}},$$

where a is firm productivity and A is an aggregator of all producing firms' productivities. $\alpha > 0$ captures the intensity of knowledge spillovers. We use a Hölder mean of the productivity of all firms producing:

$$A = \left(\frac{M_e}{M}\int_{\left(\frac{M_e}{M}\right)^{\frac{1}{\gamma}}} a^q dF(a)\right)^{\frac{1}{q}}.$$

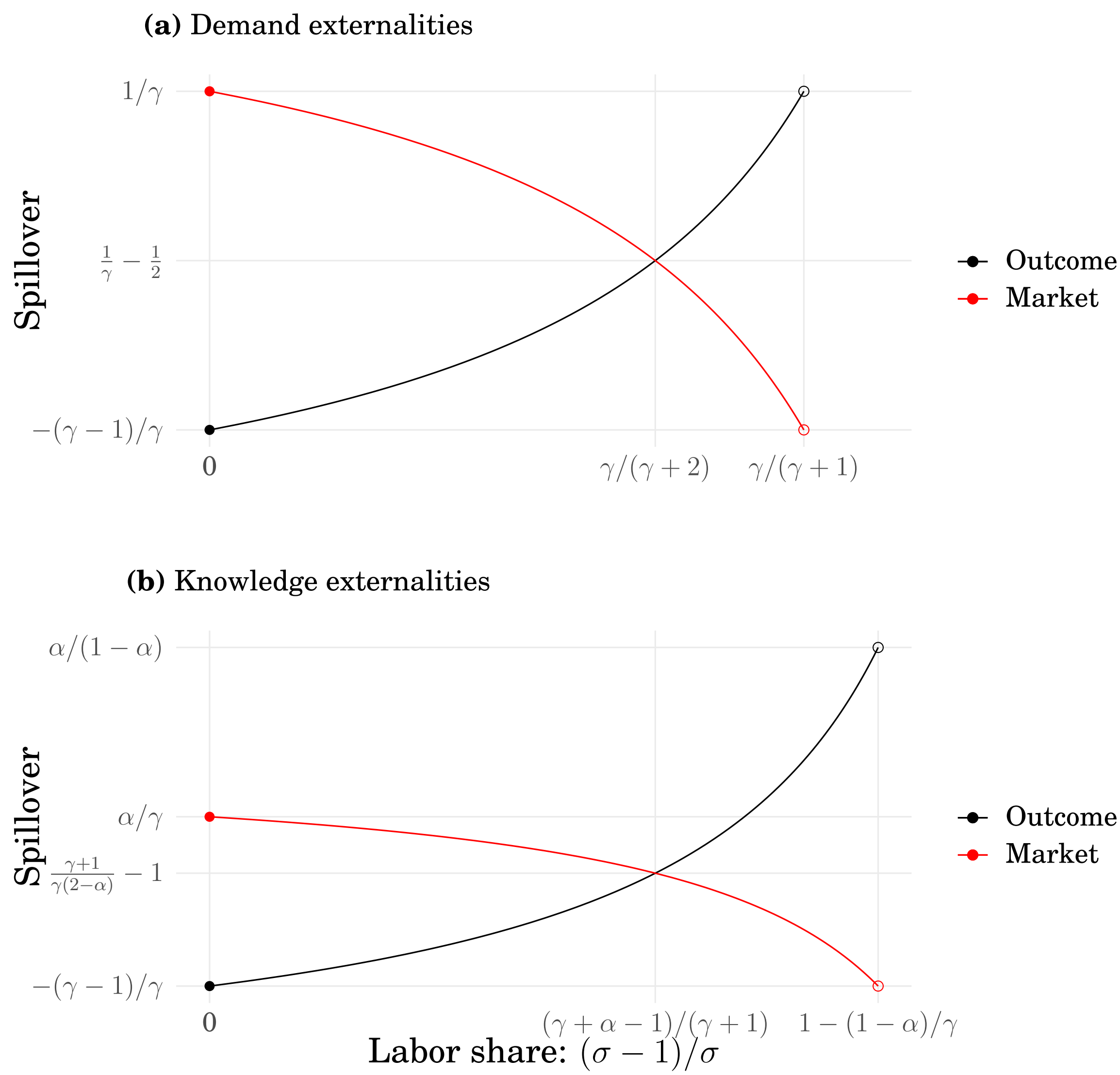

Figure IA.2
Market-based and outcome-based spillovers.

Imposing $q < \gamma$ so that the integral is well-defined, we have:

$$A = \left(\frac{\gamma}{\gamma - q}\right)^{\frac{1}{q}} \left(\frac{M_e}{M}\right)^{\frac{1}{\gamma}}.$$

Our results generalize to any aggregator that is homogeneous of degree one in the productivity distribution of producing firms. Such aggregators similarly yield an elasticity $1/\gamma$ with respect to M_e.

Firms maximize their profits, taking the wage as given:

$$\max_{\ell} \frac{\sigma}{\sigma - 1} a^{1-\alpha} A^{\alpha} \ell^{\frac{\sigma-1}{\sigma}} - w\ell.$$

The demand for labor is therefore

$$\ell = \left(\frac{w}{a^{1-\alpha}A^{\alpha}}\right)^{-\sigma},$$

and we have:

$$\begin{aligned}
y(a) &= \frac{\sigma}{\sigma-1}\left(a^{1-\alpha}A^{\alpha}\right)^{\sigma} w^{1-\sigma} \\
w\ell(a) &= \left(a^{1-\alpha}A^{\alpha}\right)^{\sigma} w^{1-\sigma} = \frac{\sigma-1}{\sigma} y(a) \\
\pi(a) &= \frac{1}{\sigma-1}\left(a^{1-\alpha}A^{\alpha}\right)^{\sigma} w^{1-\sigma} = \frac{1}{\sigma} y(a).
\end{aligned}$$

The labor share is still $(\sigma-1)/\sigma$.

The market-clearing condition for labor is:

$$\begin{aligned}
L &= w^{-\sigma} A^{\alpha\sigma} M_e \mathcal{I}_1\left(M_e, (1-\alpha)\sigma\right) \\
&= \left(\frac{\gamma}{\gamma-q}\right)^{\frac{\alpha\sigma}{q}} \left(\frac{M_e}{M}\right)^{\frac{\alpha\sigma}{\gamma}} M_e \frac{\gamma}{\gamma-(1-\alpha)\sigma} \left(\frac{M_e}{M}\right)^{\frac{(1-\alpha)\sigma}{\gamma}-1} w^{-\sigma} \\
&= \left(\frac{\gamma}{\gamma-q}\right)^{\frac{\alpha\sigma}{q}} \frac{\gamma}{\gamma-(1-\alpha)\sigma} M \left(\frac{M_e}{M}\right)^{\frac{\sigma}{\gamma}} w^{-\sigma} \\
w &= \left(\frac{M}{L}\right)^{\frac{1}{\sigma}} \left(\frac{\gamma}{\gamma-q}\right)^{\frac{\alpha}{q}} \left(\frac{\gamma}{\gamma-(1-\alpha)\sigma}\right)^{\frac{1}{\sigma}} \left(\frac{M_e}{M}\right)^{\frac{1}{\gamma}}.
\end{aligned}$$

We still have $(\sigma-1)/\sigma\mathcal{C} = wL$, and the same elasticities: $\mathcal{E}_w = \mathcal{E}_{\mathcal{C}} = \mathcal{E}_A = 1/\gamma$.

B.3.2 Spillovers

Proposition 3 and Lemma B.1 still apply, with $\mathcal{I}_1$ and $\mathcal{I}_n$ evaluated with parameter $(1-\alpha)\sigma$. The outcome-based spillover is unchanged: $spill_{out} = -(\gamma-\sigma)/\gamma$. The market-based spillover in the high-disagreement limit with $\theta > 1/\gamma$ becomes:

$$spill_{mkt}(n \to \infty) = -1 - \mathcal{E}_{\mathcal{I}_1} + \mathcal{E}_{\mathcal{C}} = -\frac{(1-\alpha)\,\sigma - 1}{\gamma}. \tag{IA.29}$$

The lower panel of Figure IA.2 shows the sign reversal of the market-based spillover, as described in Proposition 5.

C Extensions to Simple Model

C.1 Speculation with Dynamics

To capture the fact that investors' views change over time, we consider a dynamic version of the model in Section 2. The dynamic model produces predictions about the volume of trading and the length of a bubble that are consistent with documented facts in Hong and Stein (2007) and Greenwood, Shleifer, and You (2018).

We incorporate dynamics by assuming that, instead of producing at date 1, firms produce at some stochastic date T. At each date t, the economy enters the production stage with probability ψ^{-1}, and stays in the development stage with probability $1-\psi^{-1}$. At each date $t<T$ before production, firms engage into some development activity with productivity $a_{i,t}$ that is unknown. With some probability ϑ, they change their activity and receive a new productivity draw independent from their past productivity.[40] With probability $1-\vartheta$, firms' productivity stays the same as last period and $a_{i,t}=a_{i,t-1}$.

Dynamic overvaluation. In equilibrium, the price of each firm exceeds the maximum valuation of its cash flow by any specific investor in the economy. At any point in time, an investor ranking the firm first in a packet attains this maximum valuation, which we denote as $p_{i,t}^{max}$. The difference between the two valuations is

$$p_{i,t}-p_{i,t}^{max}=(\mathcal{I}_n-\mathcal{I}_1)\cdot\psi\;\frac{\vartheta(\psi-1)}{1+\vartheta(\psi-1)}>0. \tag{IA.30}$$

This difference comes from the fact that when firms change their activities, the current investor will typically not favor firm i anymore. On average, a household investing in firm i values it as a typical firm in the economy, with $\mathcal{I}_1$, rather than a favorite, with $\mathcal{I}_n$.

Predictions. The dynamic overvaluation has implications for trading volume and the length of a bubble. Each time households exchange firms signals a change in who values them most, a mechanism reminiscent of the models of Harrison and Kreps (1978) and Scheinkman and Xiong (2003). From (IA.30), we can see that the strength of the overvaluation is increasing in the volume per period ϑ and the length of the bubble ψ.[41]

C.2 Generalizing the Business-Stealing Effect

We now consider more general functions for the business-stealing effect. In particular, suppose the expected profit of a firm with productivity a is:

$$\pi(a)=a^{\eta}\delta\left(r(a,M_e)\right),$$

where $r(a,M_e)\equiv(1-F(a))\,M_e$ is the ranking of the firm, or the mass of firms with productivity greater than a. We can interpret δ as being the probability of producing conditional on a firm's ranking r. The main text focused on the special case of $\delta(r)=\mathbf{1}\{r\leq M\}$.

We continue to focus on the case with $F(a)=1-a^{-\gamma}$.

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[40] The change of firms' activity before production captures, for instance, the pivot of startups in the early stages of their development.

[41] For instance, when $\vartheta\ll\psi^{-1}\ll 1$, the overvaluation is proportional to $\vartheta\psi^2$.

C.2.1 Outcome-Based Spillover

Lemma C.1. *The outcome-based spillover does not depend on the level of entry:*

$$spill_{out} = -\frac{\gamma - \eta}{\gamma}. \tag{IA.31}$$

Proof. Under agreement, $n = 1$, and we can derive an exact solution for the mass of firms entering in equilibrium M_e. Integrating by parts, the value of a firm is:

$$\begin{aligned} V^{(1)}(M_e) &= \int_1^\infty \gamma x^{\eta-\gamma-1}\delta(M_e x^{-\gamma})dx \\ &= \frac{\gamma}{\gamma-\eta}\left[\delta(M_e) - \gamma M_e \int_1^\infty x^{\eta-2\gamma-1}\delta'(M_e x^{-\gamma})dx\right]. \end{aligned} \tag{IA.32}$$

In addition, we have:

$$M_e \frac{dV^{(1)}}{dM_e} = \gamma M_e \int_1^\infty x^{\eta-2\gamma-1}\delta'(M_e x^{-\gamma})dx. \tag{IA.33}$$

Recalling that $spill_{out} = M_e \frac{dV^{(1)}}{dM_e}/V^{(1)}$, we have the desired formula (IA.31) for the outcome-based spillover. ∎

C.2.2 Disagreement Asymptotics with Multiple Cutoffs

We now consider the generalization of δ to allow for multiple cutoffs. In particular, suppose we have cutoffs $\underline{a}_1 < ... < \underline{a}_K$, with $\underline{a}_k \equiv F^{-1}(1 - \frac{M_k}{M_e})$, and constants $\Delta_1, ..., \Delta_K$ so that

$$\delta(r) = \sum_{k=1}^{K} \Delta_k \mathbf{1}\{r \le M_k\}. \tag{IA.34}$$

Notice that this imples that $V^{(n)} = \sum_{k=1}^K \Delta_k V_k^{(n)}$, where $V_k^{(n)} \equiv \int_{\underline{a}_k}^\infty x^\eta dF_n(x)$, and

$$-M_e \frac{dV^{(n)}}{dM_e} = -\sum_{k=1}^{K}\left(\Delta_k \frac{M_k}{M_e}\cdot a^\eta \cdot \frac{F_n'}{F'}(\underline{a}_k)\right).$$

For convenience, we normalize the cost of producing blueprints so that $W(b) = f_e b^{\theta+1} M_K^{-\theta}/(\theta + 1)$.

Lemma C.2. *Holding M_e constant, the outcome-based spillover is larger than the market-based spillover.*

Proof. Apply the proof for Proposition 1 for each k. ∎

Theorem C.3 (Asymptotics for the market-based spillover with multiple cutoffs)**.** *With business-stealing of the form (IA.34), in the high-disagreement limit ($n \to \infty$), the market-based spillover converges to a finite limit.*

- *If $\gamma\theta < \eta$, then $spill_{mkt}(n) \to -(\gamma - \eta)/\gamma$.*
- *If $\gamma\theta > \eta$, then $spill_{mkt}(n) \to 0$.*

- *If* $\gamma\theta = \eta$, *then* $spill_{mkt}(n) \to -\frac{1}{f_e}\sum_{k=1}^{K}\left[\Delta_k\left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}}\alpha_\infty \exp\left(-\frac{M_k}{M_K}\alpha_\infty\right)\right]$.

Proof. Suppose $\gamma\theta < \eta$. As before, conjecture that we can write $\underline{a}_K = \alpha(n)n^{1/(\gamma(1+\theta)-\eta)}$. This yields

$$f_e \simeq \sum_{k=1}^{K}\Delta_k\left(\frac{M_k}{M_K}\right)^{\frac{\gamma-\eta}{\gamma}}\alpha^{\eta-\gamma(\theta+1)}\frac{\gamma}{\gamma-\eta}. \tag{IA.35}$$

Therefore, we have the asymptotics for firm entry:

$$\frac{M_e}{M_K} = \left[\frac{1}{f_e}\sum_{k=1}^{K}\Delta_k\left(\frac{M_k}{M_K}\right)^{\frac{\gamma-\eta}{\gamma}}\frac{\gamma}{\gamma-\eta}\cdot n\right]^{\frac{\gamma}{\gamma(\theta+1)-\eta}}. \tag{IA.36}$$

Substituting this into the formula for the market-based spillover, we have

$$spill_{mkt}(n) = -\frac{\sum_{k=1}^{K}\Delta_k\left(\frac{M_k}{M_e}\right)^{\frac{\gamma-\eta}{\gamma}} n\left(1-\frac{M_k}{M_e}\right)^{n-1}}{f_e\left(\frac{M_e}{M_K}\right)^{\theta}} \to -\frac{\gamma-\eta}{\gamma} \tag{IA.37}$$

as desired.

Now suppose $\gamma\theta > \eta$. Then we have

$$\begin{aligned} spill_{mkt}(n) &= -\frac{n\sum_{k=1}^{K}\underline{a}_k^{\eta-\gamma}\left(\underline{a}_k^{-\gamma}\right)^{n-1}}{f_e\underline{a}_K^{\gamma\theta}} \\ &= -\frac{n\sum_{k=1}^{K}\left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}}\underline{a}_K^{-\gamma}\left(1-\frac{M_k}{M_K}\underline{a}_K^{-\gamma}\right)^{n-1}}{f_e\underline{a}_K^{\gamma\theta-\eta}} \end{aligned} \tag{IA.38}$$

Suppose $\underline{a}_K \to \infty$. Then we can write the first-order condition for firm creation as:

$$\frac{f_e\underline{a}_K^{\gamma\theta-\eta}}{n\underline{a}_K^{-\gamma}} = \sum_{k=1}^{K}\Delta_k\left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}}\gamma\int_1^\infty t^{\eta-\gamma-1}\left(1-\underline{a}_k^{-\gamma}t^{-\gamma}\right)^{n-1}dt. \tag{IA.39}$$

Since the integral on the right-hand side is bounded from above by $\int_1^\infty t^{\eta-\gamma-1}dt$, we have that $n\underline{a}_K^{-\gamma} \to \infty$, which implies that we can use a similar approximation to the proof of Proposition 2 to show that:

$$spill_{mkt}(n) \simeq -\frac{n\sum_{k=1}^{K}\left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}}\underline{a}_K^{-\gamma}\exp\left(-n\frac{M_k}{M_K}\underline{a}_K^{-\gamma}\right)}{f_e\underline{a}_K^{\gamma\theta-\eta}} \to 0. \tag{IA.40}$$

Finally, suppose $\gamma\theta = \eta$. We then have

$$f_e = \sum_{k=1}^{K}\Delta_k\gamma n\left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}}\underline{a}_K^{-\gamma}\int_1^\infty t^{\eta-\gamma-1}\left(1-\frac{M_k}{M_K}\underline{a}_K^{-\gamma}t^{-\gamma}\right). \tag{IA.41}$$

As before, we conjecture that $\underline{a}_K = \alpha(n)^{-1/\gamma} n^{1/\gamma}$. Then

$$\begin{aligned}
f_e &\simeq \sum_{k=1}^{K} \Delta_k \left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \gamma\alpha(n) \int_1^\infty t^{\eta-\gamma-1} \exp\left[-(n-1)\alpha(n) n^{-1} \frac{M_k}{M_K} t^{-\gamma}\right] dt \\
&\to \sum_{k=1}^{K} \Delta_k \left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \gamma\alpha_\infty \int_1^\infty t^{\eta-\gamma-1} \exp\left[-\frac{M_k}{M_K} \alpha_\infty t^{-\gamma}\right] dt. \qquad \text{(IA.42)}
\end{aligned}$$

By analogous reasoning to the proof in Proposition 2, we can obtain a finite bound on α_∞. We therefore have the market-based spillover:

$$\begin{aligned}
spill_{mkt}(n) &= -\frac{1}{f_e} \sum_{k=1}^{K} \left[\Delta_k n \left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \underline{a}_K^{-\gamma} \left(1 - \frac{M_k}{M_K} \underline{a}_K^{-\gamma}\right)\right] \\
&\to -\frac{1}{f_e} \sum_{k=1}^{K} \left[\Delta_k \left(\frac{M_K}{M_k}\right)^{\frac{\eta-\gamma}{\gamma}} \alpha_\infty \left(1 - \frac{M_k}{M_K} \alpha_\infty\right)\right] \qquad \text{(IA.43)}
\end{aligned}$$

as desired. ▮

C.2.3 Disagreement Asymptotics with Continuous Business-Stealing

We now consider a continuous function for the business-stealing effect

$$\delta(r) = \begin{cases} 1 & \text{if } r < 1 \\ r^{-\zeta} & \text{if } r \geq 1 \end{cases} \qquad \text{(IA.44)}$$

so that ζ parameterizes the business-stealing effect for low-productivity firms with $r \geq 1$. Larger ζ implies that low-productivity firms have a lower probability of producing, with $\zeta = 0$ corresponding to the case with no business-stealing effect. With $\zeta \to \infty$, this converges to the benchmark step function business-stealing effect with $M = 1$.

We now normalize the cost of producing blueprints, so that $W(b) = f_e b^{\theta+1}/(\theta+1)$, and define $\underline{a} \equiv M_e^{\frac{1}{\gamma}}$ to be the cutoff above which $\delta(a, M_e) = 1$, i.e., firms produce with probability one.

It will be convenient to consider the decomposition $V^{(n)} = V_L^{(n)} + V_U^{(n)}$, where

$$\begin{aligned}
V_L^{(n)} &\equiv \gamma n M_e^{-\zeta} \int_1^{\underline{a}} x^{\eta-\gamma(1-\zeta)-1} (1 - x^{-\gamma})^{n-1} dx \\
V_U^{(n)} &\equiv \gamma n \int_{\underline{a}}^\infty x^{\eta-\gamma-1} (1 - x^{-\gamma})^{n-1} dx
\end{aligned}$$

capture the expected profit conditional on having productivity below and above $\underline{a}$ respectively. We can write

$$V_L^{(n)} = \gamma n \underline{a}^{\eta-\gamma(\theta+1)} \int_{\underline{a}^{-1}}^{1} t^{\eta-\gamma(1-\zeta)-1} (1 - \underline{a}^{-\gamma} t^{-\gamma})^{n-1} dt \qquad \text{(IA.45)}$$

$$V_U^{(n)} = \gamma n \underline{a}^{\eta-\gamma(\theta+1)} \int_1^{\underline{a}^{-1}} t^{\eta-\gamma-1} (1 - \underline{a}^{-\gamma} t^{-\gamma})^{n-1} dt. \qquad \text{(IA.46)}$$

Moreover, since

$$\frac{dV_L^{(n)}}{dM_e} = -\zeta M_e^{-1} V_L^{(n)} + \gamma n M_e^{-\zeta} M_e^{\frac{\eta-\gamma(1-\zeta)-1}{\gamma}} \left(1 - M_e^{-1}\right)^{n-1}$$
$$= -\zeta M_e^{-1} V_L^{(n)} - \frac{dV_U^{(n)}}{dM_e},$$

we have that

$$-M_e \frac{dV^{(n)}}{dM_e} = \zeta V_L^{(n)}. \tag{IA.47}$$

Theorem C.4 (Asymptotics for the market-based spillover with continuous business-stealing)**.** *Suppose we have business stealing of the form (IA.44) and $\zeta > \frac{\gamma-\eta}{\gamma}$. In the high-disagreement limit ($n \to \infty$), the market-based spillover converges to a finite limit.*

- *If $\gamma\theta < \eta$, then $spill_{mkt}(n) \to -(\gamma - \eta)/\gamma$.*
- *If $\gamma\theta \geq \eta$, then $spill_{mkt}(n) \to 0$.*

Proof. Suppose $\gamma\theta < \eta$. Conjecture that $\underline{a} = \alpha(n) n^{1/(\gamma(1+\theta)-\eta)}$. We have from the proof of Proposition 2 that $V_U^{(n)} \to \alpha^{\eta-\gamma}\frac{\gamma}{\gamma-\eta}$. In addition, we have from (IA.45) that $V_U^{(n)} \to \alpha^{\eta-\gamma}\frac{\gamma}{\eta-\gamma(1-\zeta)}$. Therefore, we have

$$f_e = \left(\frac{\gamma}{\gamma-\eta} - \frac{\gamma}{\gamma(1-\zeta)-\eta}\right)\alpha^{\eta-\gamma(\theta+1)}, \tag{IA.48}$$

which verifies the conjecture. We thus have the asymptotic market-based spillover

$$spill_{mkt}(n) = -\zeta \frac{V_L^{(n)}}{V^{(n)}} \to -\frac{\gamma-\eta}{\gamma} \tag{IA.49}$$

as desired.

Suppose $\gamma\theta > \eta$. Suppose $\underline{a} \to \infty$. Then we can rewrite (8) as:

$$n\underline{a}^{-\gamma} = \frac{f_e \underline{a}^{\gamma\theta-\eta}/\gamma}{\int_{\underline{a}^{-1}}^{1} t^{\eta-\gamma(1-\zeta)-1}(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1}dt + \int_1^\infty t^{\eta-\gamma-1}(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1}dt}. \tag{IA.50}$$

The two terms in the denominator of the right-hand side are bounded from above by $\int_0^1 t^{\eta-\gamma(1-\zeta)-1}dt$ and $\int_1^\infty t^{\eta-\gamma-1}dt$, respectively, which implies that $n\underline{a}^{-\gamma} \to \infty$. Using the approximation $(1-\underline{a}^{-\gamma})^{n-1} \simeq \exp(-n\underline{a}^{-\gamma})$, we have that

$$spill_{mkt}(n) = -\frac{\zeta}{f_e \underline{a}^{\gamma\theta}} V_L^{(n)} \to 0. \tag{IA.51}$$

If $\underline{a}$ has a finite limit, we can show that $V_L^{(n)} \to 0$ since $n(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1} \to 0$ for $t \in (\underline{a}^{-1}, 1)$, which implies that $spill_{mkt}(n) \to 0$ as well.

Suppose $\gamma\theta = \eta$. Using an analogous proof to Lemma A.4, we can show that $\underline{a}_n =$

$\alpha(n)^{-1/\gamma}n^{1/\gamma}$, where $\alpha(n)$ has a finite limit α_∞. Since we can bound $V_L^{(n)}$ from above by

$$V_L^{(n)} = \gamma\alpha(n)\int_{\underline{a}^{-1}}^{1} t^{\eta-\gamma(1-\zeta)-1}(1-\underline{a}^{-\gamma}t^{-\gamma})^{n-1}dt$$
$$\leq \gamma\alpha(n)\int_{\underline{a}^{-1}}^{1} t^{\eta-\gamma(1-\zeta)-1}dt \to \frac{\gamma\alpha_\infty}{\eta-\gamma(1-\zeta)},$$

we have that $spill_{mkt}(n) = -\frac{\zeta V_L^{(n)}}{f_e\underline{a}^{\gamma\theta}} \to 0.$ ▮

C.3 Results with a Zero-Cutoff-Profit Condition

Our results are robust to an extension in which the marginal firm earns zero profits. Our baseline model specifies that the M most productive firms will be allowed to produce, which allows for tractability but results in the marginal firm earning positive profits, $\pi(\underline{a}) > 0$. We now augment the model with an intermediate stage where firms, after entering the market, compete to be among one of the M firms producing. The competition stage ensures that the business-stealing externality remains. We keep the belief and production structure of the model intact= and show that the main features of the spillovers remain unchanged despite the introduction of a zero-cutoff-profit (ZCP) condition for the marginal firm.

In the new intermediate decision stage, firms can use some of their production as advertisement to reach consumers, a deadweight loss. Only the M firms that spend the most on advertising produce in equilibrium. Formally, each firm chooses how much of its production to use on advertisement, $h_i \leq \pi(a_i)$. In doing so, firms take as given the equilibrium level $\underline{h}$ of advertising necessary to attract consumers. Their profit function is therefore $\pi(a_i)\mathbf{1}\{h_i \geq \underline{h}\} - h_i$. The optimal advertisement choice is $h_i = \underline{h}$ if $\pi(a_i) \geq \underline{h}$ and 0 otherwise. The equilibrium value of $\underline{h}$ is such that exactly M firms choose to spend on advertisement. Keeping the definition of the production cutoff $\underline{a}$ from earlier, this implies

$$\underline{h} = \pi(\underline{a}). \tag{IA.52}$$

Firms must spend the profits of the marginal firm to be able to produce, resulting in zero profits for the marginal firm.

C.3.1 General Derivations

Firm value in this model is modified to account for the cost of advertisement:

$$\widetilde{V}^{(n)}(M_e) = \int_{\underline{a}}^{\infty} (\pi(a) - \pi(\underline{a}))\, dF^n(a). \tag{IA.53}$$

We can define the corresponding integral $\widetilde{\mathcal{I}}_n$. With this new definition of firm value, the remainder of the competitive equilibrium and the planner problem are unchanged. In particular, the market-based spillover is $spill_{mkt} = \mathcal{E}_{\widetilde{\mathcal{I}}_n}$.

Decompose firms' valuations into the revenue (from (IA.53)) and advertising cost components:

$$V^{(n)}(M_e) = \int_{F^{-1}\left(1-\frac{M}{M_e}\right)}^{\infty} \pi(a)dF_n(a) - \left(\frac{M}{M_e}\right)^{1-\frac{\eta}{\gamma}} \cdot n\left(1-\frac{M}{M_e}\right)^{n-1}. \tag{IA.54}$$

The first derivative of $V^{(n)}$ is:

$$-M_e \cdot \frac{dV^{(n)}(M_e)}{dM_e} = \frac{\eta}{\gamma}\left(\frac{M_e}{M}\right)^{\frac{\eta}{\gamma}} \cdot \left[1-\left(1-\frac{M}{M_e}\right)^n\right].$$

Using the free-entry condition, $V^{(n)}(M_e) = W'(M_e)$, we have following formula for the market-based spillover:

$$spill_{mkt}(n; M_e) = \frac{\eta}{\gamma} \cdot \frac{1}{f_e} \cdot \left(\frac{M_e}{M}\right)^{\frac{\eta}{\gamma}-\theta} \cdot \left[1-\left(1-\frac{M}{M_e}\right)^n\right]. \tag{IA.55}$$

C.3.2 Spillovers and Firm Entry

Lemma C.5. *In the model with a ZCP condition, the outcome-based spillover is:*

$$spill_{out} = -\frac{\gamma-\eta}{\gamma}.$$

Proof. The free-entry condition with $n=1$ gives us:

$$\left(\frac{M_e}{M}\right)^{\frac{\gamma(1+\theta)-\eta}{\gamma}} = \frac{1}{f_e} \cdot \frac{\eta}{\gamma-\eta}.$$

Given the derivation of the spillover in (IA.55), we have:

$$spill_{out}(M_e) = -\frac{\eta}{\gamma} \cdot \frac{1}{f_e} \cdot \left(\frac{M_e}{M}\right)^{\frac{\eta}{\gamma}-\theta} \cdot \frac{M}{M_e} = -\frac{\gamma-\eta}{\gamma}, \tag{IA.56}$$

where we have used our equilibrium solution for M_e/M. ∎

Lemma C.6. *In the high-disagreement limit ($n \to \infty$), the mass of entrants also increases and goes to infinity:* $\lim_{n\to\infty} M_e = \infty$.

Proof. We adapt the proof from Lemma A.3, again defining the sequence $\underline{a}_n = (M_e/M)^{1/\gamma}$ and showing that $\underline{a}_n \to \infty$. Equation (IA.54) implies the implicit definition of the sequence $(\underline{a}_n)_n$ in this case:

$$f_e \underline{a}_n^{\gamma(\theta+1)-\eta} = \gamma n \int_1^{\infty} t^{\eta-\gamma-1}\left(1-\underline{a}_n^{-\gamma}t^{-\gamma}\right)^{n-1} dt - n(1-\underline{a}_n^{-\gamma})^{n-1}.$$

Assume that $\underline{a}_n$ has a finite limit that is strictly larger than zero, $\underline{a}_\infty > 0$. Then there exists N large enough such that $\forall n > N, \underline{a}_n > A = \underline{a}_\infty - \epsilon > 0$. For any arbitrary threshold T_n, we have

$$I_n > n\left[\frac{\gamma}{\gamma-\eta} \cdot T_n^{\eta-\gamma}\left(1-A^{-\gamma}T_n^{-\gamma}\right)^{n-1} - 1\right].$$

As in Lemma A.3, we conclude by considering the threshold $T_n = n^{1/\gamma}$. ∎

Lemma C.7 (Asymptotics for firm creation)**.** *In the high-disagreement limit ($n \to \infty$), we have the following asymptotics for the mass of firms created M_e:*

- *If $\gamma\theta < \eta$, then $\lim_{n\to\infty} M_e/M = \alpha_\infty^\gamma n^{\frac{\gamma}{\gamma(1+\theta)-\eta}}$.*
- *If $\gamma\theta = \eta$, then $\lim_{n\to\infty} M_e/M = \alpha_\infty^{-1} n$.*

In each case, α_∞ is a constant defined below.

Proof. We adapt the proof from Lemma A.4. Starting from (8), and using $\underline{a}$:

$$f_e \simeq \gamma \underline{a}^{\eta-\gamma(\theta+1)} n \int_1^\infty \left(t^\eta - 1\right) t^{-\gamma-1} \exp\left(-(n-1)\underline{a}^{-\gamma} t^{-\gamma}\right) dt,$$

where we have used the fact that $\underline{a} \to \infty$ and $\log(1-x) = -x + \mathcal{O}(x^2)$. To find a solution, we guess that asymptotically $\underline{a} \simeq \alpha(n) n^{\frac{1}{\gamma(1+\theta)-\eta}}$ and show that $\alpha(n)$ converges to a finite limit α. The first-order condition becomes

$$f_e \simeq \gamma \alpha(n)^{\eta-\gamma(\theta+1)} \int_1^\infty \left(t^\eta - 1\right) t^{-\gamma-1} \exp\left(-\alpha(n)^{-\gamma} \frac{n-1}{n^{\frac{\gamma}{\gamma(1+\theta)-\eta}}} t^{-\gamma}\right) dt.$$

If $\gamma\theta < \eta$, then the exponential term converges to zero and we have:

$$\alpha_\infty = \left(\frac{1}{f_e} \cdot \frac{\eta}{\gamma - \eta}\right)^{\frac{1}{\gamma(1+\theta)-\eta}}. \tag{IA.57}$$

If $\gamma\theta = \eta$, then $\underline{a}$ is defined by:

$$f_e = \gamma \underline{a}_n^{-\gamma} n \int_1^\infty t^{\eta-\gamma-1} \left(1 - \underline{a}^{-\gamma} t^{-\gamma}\right)^{n-1} dt.$$

We guess and verify that $\underline{a}_n = \alpha(n)^{-1/\gamma} n^{1/\gamma}$, and $\alpha(n)$ has a finite limit α_∞:

$$f_e = \gamma \alpha_\infty \int_1^\infty \left(t^\eta - 1\right) t^{-\gamma-1} e^{-\alpha_\infty t^{-\gamma}} dt,$$

where we took the limit when $n \to \infty$. Moreover, we are able to bound the magnitude of the market-based spillover above using a bound on α_∞:

$$\frac{\alpha_\infty e^{-\alpha_\infty}}{f_e} < \frac{\gamma - \eta}{\eta}, \tag{IA.58}$$

which verifies that α_∞ is finite. ∎

We now show that, despite the presence of the ZCP, Proposition 2 holds.

Theorem C.8. *In the high-disagreement limit ($n \to \infty$), the market-based spillover converges to a finite limit.*

- *If $\gamma\theta < \eta$, then $spill_{mkt}(n) \to -(\gamma - \eta)/\gamma$.*
- *If $\gamma\theta > \eta$, then $spill_{mkt}(n) \to 0$.*
- *If $\gamma\theta = \eta$, then $spill_{mkt}(n) \to -\frac{\eta}{\gamma}\frac{1}{f_e} e^{-\alpha_\infty}$.*

Proof. If $\gamma\theta > \eta$, then given equation (IA.55), we use that $M_e \to \infty$ to conclude that $\lim_{n\to\infty} spill_{mkt} = 0$.

If $\gamma\theta < \eta$, then we can use the asymptotics from C.7 and the formula for the market-based spillover from (IA.55):

$$spill_{mkt}(n; M_e) \simeq -\frac{\eta}{\gamma}\frac{1}{f_e}\alpha_\infty^{\eta-\theta\gamma} \cdot n^{\frac{\eta-\theta\gamma}{\gamma(1+\theta)-\eta}} \cdot \left[1 - \left(1 - \alpha_\infty^{-\gamma} n^{\frac{-\gamma}{\gamma(1+\theta)-\eta}}\right)^n\right] \tag{IA.59}$$

$$\simeq -\frac{\eta}{\gamma}\frac{1}{f_e}\alpha_\infty^{\eta-\theta\gamma} \cdot n^{\frac{\eta-\theta\gamma}{\gamma(1+\theta)-\eta}} \cdot \left[1 - \exp\left(-\alpha_\infty^{-\gamma} n^{\frac{\gamma\theta-\eta}{\gamma(1+\theta)-\eta}}\right)\right] \tag{IA.60}$$

$$\simeq -\frac{\eta}{\gamma}\frac{1}{f_e}\alpha_\infty^{\eta-\theta\gamma} \cdot n^{\frac{\eta-\theta\gamma}{\gamma(1+\theta)-\eta}} \cdot \alpha_\infty^{-\gamma} n^{\frac{\gamma\theta-\eta}{\gamma(1+\theta)-\eta}} \to -\frac{\eta}{\gamma}\frac{1}{f_e}\alpha_\infty^{\eta-\gamma(1+\theta)}. \tag{IA.61}$$

Using the definition of α_∞ from the proof above, we conclude $\lim_{n\to\infty} spill_{mkt}(n; M_e) = -(\gamma - \eta)/\gamma$.

In the knife-edge case with $\gamma\theta = \eta$, we have

$$spill_{mkt}(n; M_e) \simeq -\frac{\eta}{\gamma}\frac{1}{f_e} \cdot \left[1 - \left(1 - \alpha_\infty n^{-1}\right)^n\right] \to -\frac{\eta}{\gamma}\frac{1}{f_e} \cdot e^{-\alpha_\infty}. \tag{IA.62}$$

We can bound the market-based spillover in the limit: $\lim_{n\to\infty} |spill_{mkt}(n; M_e)| < \alpha_\infty^{-1} \cdot (\gamma-\eta)/\gamma$. ∎

Our conclusions are therefore robust to including competition to enter. Intuitively, marginal firms drive the externality in both settings. In our baseline, the externality operates at the extensive margin: more entry displaces the profits of excluded marginal firms. In this model, the externality is on the intensive margin: firm entry increases the productivity of the marginal firm and thus advertisement costs for all producing firms.

D Extensions to General Equilibrium Model

D.1 Elastic Labor Supply

D.1.1 Setting and Equilibrium

We now consider the case of variable labor supply. Households can provide labor L by exerting an effort cost $S(L)$. We assume

$$S'(L) = f_l \left(\frac{L}{L_0}\right)^{1/\kappa},$$

where κ is the Frisch elasticity of labor supply. As κ converges to 0, the model converges to a constant labor supply L_0. The remainder of the model is unchanged.

In equilibrium, we have $S'(L) = w$, which implies that $\mathcal{E}_L = \kappa\mathcal{E}_w$. Noting that we still have a constant labor share ($(\sigma-1)/\sigma\mathcal{C} = wL$), we also have $\mathcal{E}_C = \mathcal{E}_w + \mathcal{E}_L = (1+\kappa)\mathcal{E}_w$.

Market clearing for labor yields:

$$L = M_e \cdot w^{-\sigma} \int_{\underline{a}}^{\infty} a^{-\sigma} dF(a) = M_e \cdot w^{-\sigma} \cdot \mathcal{I}_1,$$

which given the expression for $\mathcal{I}_1$ in (IA.23) under a Pareto leads to the following restriction:

$$wL^{1/\sigma} = \left(\frac{\gamma}{\gamma-\sigma}\right)^{\frac{1}{\sigma}} \cdot M_e^{\frac{1}{\sigma}} \cdot \left(\frac{M_e}{M}\right)^{\frac{1}{\gamma}-\frac{1}{\sigma}}.$$

Using the labor supply equation, we obtain

$$\mathcal{E}_w = \frac{1}{\gamma}\frac{\sigma}{\sigma+\kappa}$$
$$\mathcal{E}_\mathcal{C} = \frac{1}{\gamma}\frac{\sigma+\kappa\sigma}{\sigma+\kappa}.$$

As we increase the elasticity of labor supply, the wage becomes less elastic to firm entry and consumption becomes more elastic to firm entry as it becomes less costly to expand labor.

D.1.2 Spillovers

Asymptotics. To study the asymptotic behavior of the market-based spillover, recall the free-entry condition:

$$f_e \left(\frac{M_e}{M}\right)^{\theta} = \frac{1}{\sigma-1} w^{1-\sigma} \mathcal{I}_n.$$

We define

$$\tilde{\theta} = \theta + \frac{\sigma-1}{\gamma}\frac{\sigma}{\kappa+\sigma} \geq 0$$

and recognize the free-entry condition from the pure business-stealing model. This guarantees that $\lim_{n\to\infty} \mathcal{I}_1/\mathcal{I}_n = 0$. In addition, $\mathcal{E}_{\mathcal{I}_n}$ converges to 0 if:

$$\gamma\tilde{\theta} > \sigma \tag{IA.63}$$

$$\Leftrightarrow \gamma\theta > 1 + (\sigma - 1)\frac{\kappa}{\kappa + \sigma}. \tag{IA.64}$$

Behavior of the spillovers. The market-based social value is characterized by:

$$\begin{aligned} & \frac{1}{\sigma}\mathcal{C}\frac{\mathcal{I}_n}{\mathcal{I}_1} + \frac{\sigma-1}{\sigma}\mathcal{C} - S(L) - W(M_e) \\ = \quad & \frac{1}{\sigma}\mathcal{C}\frac{\mathcal{I}_n}{\mathcal{I}_1}\frac{\mathcal{I}_n'}{\mathcal{I}_n} + \frac{1}{\sigma}\mathcal{C}'\frac{\mathcal{I}_n}{\mathcal{I}_1} - \frac{1}{\sigma}\mathcal{C}\frac{\mathcal{I}_n}{\mathcal{I}_1}\frac{\mathcal{I}_1'}{\mathcal{I}_1} + \frac{\sigma-1}{\sigma}\mathcal{C}' - \underbrace{S'(L)L}_{\frac{\sigma-1}{\sigma}\mathcal{C}}\frac{1}{L}\frac{dL}{dM_e}. \end{aligned}$$

The market-based spillover is similar to Proposition 3. However, the appropriability term now accounts for the utility cost of expanding the labor supply:

$$spill_{mkt}(n) = \mathcal{E}_{\mathcal{I}_n} - \mathcal{E}_{\mathcal{I}_1} - 1 + \mathcal{E}_{\mathcal{C}} + (\sigma - 1)\underbrace{(\mathcal{E}_{\mathcal{C}} - \mathcal{E}_L)}_{\mathcal{E}_w}\frac{\mathcal{I}_1}{\mathcal{I}_n}.$$

The outcome-based spillover is:

$$spill_{out} = -1 + \mathcal{E}_{\mathcal{C}} + (\sigma - 1)\mathcal{E}_w = -\frac{\gamma - \sigma}{\gamma}. \tag{IA.65}$$

With high speculation and large θ, we have:

$$spill_{mkt}(n \to \infty) = -(\sigma - 1)\mathcal{E}_w = -\frac{\sigma - 1}{\gamma}\frac{\sigma}{\kappa + \sigma}. \tag{IA.66}$$

The outcome-based spillover is unchanged from the baseline model. Thee market-based spillover with high disagreement is decreasing in σ and increasing in γ, as is the case in Proposition 4. The elasticity of labor supply does not affect the outcome-based spillover, but under market-based measures, increasing κ increases the spillover. As labor supply becomes more elastic, the wage becomes less responsive to entry, and firms have less influence on each other through general equlibrium effects. With perfectly elastic labor supply, there are no general equilibrium effects and hence no market-based spillovers in the high-disagreement limit, i.e., $spill_{mkt}(n \to \infty) = 0$.

D.2 Variable Number of Participating Firms

D.2.1 Setting and Equilibrium

We study a model where the number of participating firms, M, responds to firm creation M_e, which can be interpreted as households' consumption bundles becoming more or less concentrated as more firms enter the economy. We assume that M varies exogenously with the level of firm entry M_e:

$$M = \frac{1}{M_0^{\chi-1}} \cdot M_e^{\chi},$$

where χ is the elasticity of firms producing to firms created and M_0 a normalization constant. We assume that $\chi \leq 1$ such that we always have $M \leq M_e$.

The cost of creating a firm only depends on M_0 through:

$$W'(M_e) = f_e \left(\frac{M_e}{M_0}\right)^{\theta}.$$

The productivity threshold to produce is now:

$$\underline{a} := F^{-1}\left(1 - \frac{M}{M_e}\right) = \left(\frac{M_e}{M_0}\right)^{\frac{1-\chi}{\gamma}}.$$

The model still features a constant labor share and firm profits are still isoelastic in the productivity:

$$\pi(a) = \frac{1}{\sigma - 1} \cdot w^{1-\sigma} \cdot a^{\sigma} = \frac{1}{\sigma}\frac{\mathcal{C}}{M_e} \cdot \frac{a^{\sigma}}{\mathcal{I}_1},$$

where we have redefined the integrals $\mathcal{I}_1$ and $\mathcal{I}_n$ to adjust for the new expressions for the productivity threshold $\underline{a}$:

$$\mathcal{I}_n(\chi) = \int_{\left(\frac{M_e}{M_0}\right)^{\frac{1-\chi}{\gamma}}}^{\infty} a^{\sigma} dF_n(a)$$

$$\mathcal{I}_1(\chi) = \frac{\gamma}{\gamma - \sigma} \cdot \left(\frac{M_e}{M_0}\right)^{(\chi-1)\frac{\gamma-\sigma}{\gamma}}.$$

The market-clearing condition $L = M_e w^{-\sigma} \mathcal{I}_1$ implies the equilibrium wage:

$$w = \left(\frac{\gamma}{\gamma - \sigma}\right)^{\frac{1}{\sigma}} \cdot L^{-\frac{1}{\sigma}} M_0^{(1-\chi)\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)} \cdot M_e^{\frac{\chi}{\sigma}+\frac{1-\chi}{\gamma}},$$

so that the labor elasticity is :

$$\mathcal{E}_w = \frac{1}{\gamma} + \chi \cdot \left(\frac{1}{\sigma} - \frac{1}{\gamma}\right).$$

We obtain aggregate consumption by aggregating individual output $\mathcal{C} = M_e\sigma/(\sigma - 1)w^{1-\sigma}\mathcal{I}_1$, which yields equilibrium aggregate consumption and elasticity:

$$\mathcal{C} = \frac{\sigma}{\sigma - 1}\frac{\gamma}{\gamma - \sigma} \cdot L^{\frac{\sigma-1}{\sigma}} \cdot M_0^{(1-\chi)\left(\frac{1-\sigma}{\sigma}+\frac{\gamma-1}{\gamma}\right)} \cdot M_e^{\frac{1}{\gamma}+\chi\cdot\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)}$$

$$\mathcal{E}_{\mathcal{C}} = \mathcal{E}_w = \frac{1}{\gamma} + \chi \cdot \left(\frac{1}{\sigma} - \frac{1}{\gamma}\right)$$

D.2.2 Spillovers

Given the constant labor share and isoelastic profits, we can apply Proposition 3 and obtain the market-based spillover:

$$spill_{mkt}(n) = \mathcal{E}_{\mathcal{I}_n(\chi)} - 1 - \mathcal{E}_{\mathcal{I}_1(\chi)} - \mathcal{E}_{\mathcal{C}} + (\sigma - 1)\mathcal{E}_C \cdot \frac{\mathcal{I}_1(\chi)}{\mathcal{I}_n(\chi)}. \tag{IA.67}$$

From the expression for $\mathcal{I}_n$ we have the following change in the elasticities:

$$\mathcal{E}_{\mathcal{I}_n(\chi)} = (1-\chi)\mathcal{E}_{\mathcal{I}_n(\chi=0)} = (1-\chi)\mathcal{E}_{\mathcal{I}_n} \tag{IA.68}$$

Asymptotics. We now turn to the high-disagreement limit. The first-order condition for firm creation is:

$$\begin{aligned} & f_e\left(\frac{M_e}{M_0}\right)^{\theta} = \frac{1}{\sigma-1} w^{1-\sigma}\mathcal{I}_n(\chi) \\ \iff \quad & \text{constant} = \underline{a}^{-\theta\frac{\gamma}{1-\chi}+(1-\sigma)\frac{\gamma}{1-\chi}\left(\frac{1}{\gamma}+\chi\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)\right)} \int_{\underline{a}}^{\infty} a^{\sigma} dF_n(a). \end{aligned}$$

We define:

$$\tilde{\theta} = \frac{1}{1-\chi}\left(\theta + \frac{\sigma-1}{\gamma} + \chi(\sigma-1)\left(\frac{1}{\sigma} - \frac{1}{\gamma}\right)\right),$$

and recognize the entry condition of the baseline model. We apply our previous results, changing the condition for $\mathcal{E}_{\mathcal{I}_n(\chi)} \to 0$ to $\gamma\tilde{\theta} > \sigma$, which reduces to:

$$\gamma(\theta+\chi) > 1 + \chi\left(\frac{\gamma}{\sigma} - 1\right). \tag{IA.69}$$

If this condition is satisfied, then $\lim_{n\to\infty} \mathcal{E}_{\mathcal{I}_n(\chi)} = 0$. When the inequality is reversed, $\lim_{n\to\infty} \mathcal{E}_{\mathcal{I}_n(\chi)} = \mathcal{E}_{\mathcal{I}_1(\chi)} = (1-\chi)(\sigma-\gamma)/\gamma$. With equality, the elasticity admits a finite limit between these two values.

Behavior of the spillovers. The outcome-based spillover is

$$spill_{out} = -1 + \sigma\mathcal{E}_{\mathcal{C}} = -\frac{\gamma-\sigma}{\gamma} + \chi\sigma\left(\frac{1}{\sigma} + \frac{1}{\gamma}\right). \tag{IA.70}$$

The market-based spillover with high disagreement, when θ is large enough, is:

$$spill_{mkt}(n\to\infty) = -1 - \mathcal{E}_{\mathcal{I}_1(\chi)} + \mathcal{E}_{\mathcal{C}} = -\frac{\sigma-1}{\gamma} - \chi(\sigma-1)\left(\frac{1}{\sigma} - \frac{1}{\gamma}\right) \tag{IA.71}$$

As long as $|\chi| < 1$, Proposition 4 still holds with the generalized version of $\mathcal{I}_n$. A higher elasticity of firm participation with respect to firm entry leads to opposite results with and without disagreement. A large elasticity χ dampens the outcome-based spillovers because it diminishes business stealing. However it strengthens the market-based spillover with disagreement: in response to firm entry, labor demand responds at the intensive margin with more productive firms and at the extensive margin with more participating firms.

D.3 Advertising to Participate

We augment the previous model with an intermediate stage after market entry when firms compete to be one of M firms producing, as in Appendix C.3.

D.3.1 Setting and Equilibrium

In the new intermediate decision stage, firms choose to spend on advertisement to reach consumers. We assume that only the M firms that spend the most on advertising produce in equilibrium. Formally if a firm with productivity a_i spends h_i in advertising, its profit is: $\pi(a_i)\mathbf{1}\{h_i \geq \underline{h}\} - h_i$. Hence there is a threshold level of advertising, $\underline{h}$, below which firms cannot reach any consumers and above which firms do produce. Firms take the threshold as given and decide on their choice of advertising. Thus the advertising equilibrium is such that the threshold matches the profit of the marginal firm: $\underline{h} = \pi(\underline{a})$.

Profits are modified with respect to the standard model of Section B.1 to incorporate the advertisement payments:

$$\pi(a) = \frac{1}{\sigma} w^{1-\sigma} \left(a^{\sigma} - \underline{a}^{\sigma}\right),$$

The ex-ante firm valuation is therefore:

$$V^{(n)}(M_e) = \frac{1}{\sigma} \cdot \frac{\mathcal{C}}{M_e} \cdot \frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1},$$

where we define the adjusted integral $\widetilde{\mathcal{I}}_n$ as

$$\widetilde{\mathcal{I}}_n(M_e) = \int_{\left(\frac{M_e}{M}\right)^{\frac{1}{\gamma}}}^{\infty} \left(a^{\sigma} - \left(\frac{M_e}{M}\right)^{\frac{\sigma}{\gamma}}\right) dF_n(a).$$

The entry condition in the competitive equilibrium is now:

$$W'(M_e) = \frac{1}{\sigma} \cdot \frac{\mathcal{C}}{M_e} \cdot \frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}.$$

D.3.2 Spillovers

The market-based social value is characterized by:

$$\frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}\frac{\widetilde{\mathcal{I}}'_n}{\widetilde{\mathcal{I}}_n} + \frac{1}{\sigma}\mathcal{C}'\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1} - \frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}\frac{\mathcal{I}'_1}{\mathcal{I}_1} + \frac{\sigma-1}{\sigma}\mathcal{C}'.$$

The market-based spillover is therefore:

$$spill_{mkt}(n) = \mathcal{E}_{\widetilde{\mathcal{I}}_n} - \mathcal{E}_{\mathcal{I}_1} - 1 + \mathcal{E}_{\mathcal{C}} + (\sigma-1)\mathcal{E}_{\mathcal{C}}\frac{\mathcal{I}_1}{\widetilde{\mathcal{I}}_n}. \tag{IA.72}$$

The outcome-based spillover (or market-based spillover under agreement) is:

$$spill_{out} = -1 + \mathcal{E}_{\mathcal{C}} + (\sigma-1)\mathcal{E}_{\mathcal{C}}\frac{\gamma}{\sigma} = -\frac{1}{\sigma} + \frac{1}{\gamma}, \tag{IA.73}$$

where we have used that, as in the baseline model, $\mathcal{E}_{\mathcal{C}} = \mathcal{E}_w = 1/\gamma$. However, the profit share is now lower because of advertisement costs. A fraction $(\sigma-1)/\sigma$ of output accrues to labor but only a fraction $1/\gamma$ ($< 1/\sigma$) is collected as profits. The larger importance of labor relative to profits gives rise to a smaller outcome-based spillover. As in the standard model, $|spill_{out}|$ is decreasing in σ and increasing in γ.

To derive the market-based spillover with high disagreement, notice that $\mathcal{I}_1/\widetilde{\mathcal{I}}_n \to 0$ and therefore, if $\theta > 1/\gamma$, then $spill_{mkt}(n \to \infty)$ is unchanged from the standard model:

$$spill_{mkt}(n \to \infty) = -\frac{\sigma-1}{\gamma}. \tag{IA.74}$$

Therefore, Proposition 4 still holds.

D.3.3 Advertising with Demand or Knowledge Spillovers.

Introducing advertisement costs to the aggregate demand and knowledge externalities models from Sections B.2 and B.3 yields the same formula for $spill_{mkt}(n)$ as in equation (IA.72), so that we have the limit with high disagreement $spill_{mkt}(n \to \infty) = -(\sigma-1)/\gamma$.

The main difference between spillovers in the standard model and the models in Section B.2 and B.3 arises through the differences in the profit function, which affects the ratio $\mathcal{I}_1/\widetilde{\mathcal{I}}_n$. With aggregate demand externalities, the ratio is $\gamma/(\sigma-1)$, and the outcome-based spillover is:

$$spill_{out}^{\text{AD}} = \frac{1}{\gamma}. \tag{IA.75}$$

For the model with knowledge spillovers, the ratio is $\gamma/(1-\alpha)\sigma$, and the outcome-based spillover is:

$$spill_{out}^{\text{KS}} = -1 + \frac{1}{\gamma} + \frac{\sigma-1}{(1-\alpha)\sigma}. \tag{IA.76}$$

D.4 Participation Costs in the Baseline Model

Another way to ensure the marginal firm makes zero profit is to assume firms invest in infrastructure to produce. In particular, suppose that upon entry all firms can participate on the goods market, but firms must buy one unit of infrastructure to reach all of their customers. Households produce infrastructure competitively at a cost of effort Φ. In an equilibrium with M producing firms, the price of infrastructure is:

$$\Phi'(M) = \varphi(M) = \varphi_0 \cdot M^{\nu}$$

with $\nu > 0$, so that the cost of infrastructure is increasing in the mass of producing firms M.

D.4.1 Setting and Equilibrium

Participating firms. Given M_e and M, profits before the infrastructure costs are unchanged from the standard model with decreasing returns to scale:

$$\pi(a) = \frac{1}{\sigma-1} \cdot w^{1-\sigma} \cdot a^{\sigma}.$$

The equilibrium wage is also unchanged:

$$w = \left(\frac{\gamma}{\gamma - \sigma}\right)^{\frac{1}{\sigma}} \cdot \left(\frac{M_e}{L}\right)^{\frac{1}{\sigma}} \cdot \left(\frac{M_e}{M}\right)^{\frac{1}{\gamma} - \frac{1}{\sigma}}.$$

The marginal firm has productivity $\underline{a}$ and spends all of its profit on infrastructure. Therefore, we have the zero-cutoff-profit condition $\Phi'(M) = \pi(\underline{a})$, which implies:

$$M^{\nu + \frac{1}{\gamma} + \frac{\sigma-1}{\sigma}} = \frac{1}{\varphi_0} \frac{1}{\sigma - 1} \left(\frac{\gamma - \sigma}{\gamma}\right)^{\frac{\sigma-1}{\sigma}} L^{\frac{\sigma-1}{\sigma}} \cdot M_e^{\frac{1}{\gamma}},$$

where we use the fact that $\underline{a} = (M_e/M)^{1/\gamma}$. In Section D.2, we specified an exogenous set of producing firms $M = M_e^{\chi}/M_0^{\chi-1}$. This arises endogenously through our cost of infrastructure with

$$\chi = \frac{1}{\gamma}\left(\nu + \frac{1}{\gamma} + \frac{\sigma - 1}{\sigma}\right)^{-1}$$

$$M_0^{1-\chi} = \left(\frac{1}{\varphi_0} \frac{1}{\sigma - 1} \left(\frac{\gamma - \sigma}{\gamma}\right)^{\frac{\sigma-1}{\sigma}} L^{\frac{\sigma-1}{\sigma}}\right)^{\left(\nu + \frac{1}{\gamma} + \frac{\sigma-1}{\sigma}\right)^{-1}},$$

where the exponent satisfies $\chi \leq 1$.

We can also compute the elasticity $\mathcal{E}_{\mathcal{C}}$:

$$\mathcal{E}_{\mathcal{C}} = \frac{1}{\gamma} + \chi\left(\frac{1}{\sigma} - \frac{1}{\gamma}\right) = \frac{1}{\gamma} \cdot \frac{1 + \nu}{1 + \nu + 1/\gamma - 1/\sigma} = \chi \cdot (1 + \nu).$$

Equilibrium. The equilibrium condition in the competitive equilibrium is:

$$W'(M_e) = \frac{1}{\sigma} \cdot \frac{\mathcal{C}}{M_e} \cdot \frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1},$$

where we define the modified $\widetilde{\mathcal{I}}_n$ integral to account for the infrastructure expenditures of the firm:

$$\widetilde{\mathcal{I}}_n(M_e, \chi) = \int_{\left(\frac{M_e}{M_0}\right)^{\frac{1-\chi}{\gamma}}}^{\infty} \left(a^{\sigma} - \left(\frac{M_e}{M_0}\right)^{\sigma \frac{1-\chi}{\gamma}}\right) dF_n(a).$$

With $n = 1$, we have:

$$\widetilde{\mathcal{I}}_1(M_e, \chi) = \frac{\sigma}{\gamma - \sigma} \cdot \left(\frac{M_0}{M_e}\right)^{(1-\chi)\frac{\gamma-\sigma}{\gamma}} = \frac{\sigma}{\gamma} \cdot \mathcal{I}_1(M_e, \chi).$$

Aggregate profits therefore represent a fraction σ/γ of aggregate revenue after labor costs, while aggregate infrastructure costs account for the other $(\gamma - \sigma)/\gamma$. Therefore, aggregate profits represent a share $1/\gamma$ of consumption and aggregate infrastructure costs $1/\sigma - 1/\gamma$.

D.4.2 Spillovers

The market-based social value is now characterized by:

$$\frac{d}{dM_e}\left[\frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}+\frac{\sigma-1}{\sigma}\mathcal{C}+\underbrace{\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)\mathcal{C}}_{\substack{\text{consumption from}\\ \text{infrastructure}}}-\underbrace{\Phi(M)}_{\substack{\text{cost of}\\ \text{infrastructure}}}\right]$$

$$=\frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}\frac{\widetilde{\mathcal{I}}_n'}{\widetilde{\mathcal{I}}_n}+\frac{1}{\sigma}\mathcal{C}'\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}-\frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}\frac{\mathcal{I}_1'}{\mathcal{I}_1}+\frac{\sigma-1}{\sigma}\mathcal{C}'+\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)\mathcal{C}'-\underbrace{\Phi'(M)M}_{\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right)\mathcal{C}}\cdot\frac{1}{M}\frac{dM}{dM_e}.$$

The market-based spillover is therefore:

$$spill_{mkt}(n)=\mathcal{E}_{\widetilde{\mathcal{I}}_n}-\mathcal{E}_{\mathcal{I}_1}-1+\mathcal{E}_{\mathcal{C}}+(\sigma-1)\mathcal{E}_{\mathcal{C}}\cdot\frac{\mathcal{I}_1}{\widetilde{\mathcal{I}}_n}+\underbrace{\frac{\gamma-\sigma}{\gamma}(\mathcal{E}_{\mathcal{C}}-\mathcal{E}_M)\cdot\frac{\mathcal{I}_1}{\widetilde{\mathcal{I}}_n}}_{\substack{\text{surplus from}\\ \text{infrastructure costs}}},$$

where the last term accounts for the surplus from infrastructure creation.

The outcome-based spillover (and market-based spillover under agreement) is:

$$spill_{out}=-1+\mathcal{E}_{\mathcal{C}}+\left[(\sigma-1)\mathcal{E}_{\mathcal{C}}+\left(1-\frac{\sigma}{\gamma}\right)(\mathcal{E}_{\mathcal{C}}-\chi)\right]\frac{\gamma}{\sigma}.$$

Using the values of $\mathcal{E}_{\mathcal{C}}$ and χ, we obtain:

$$spill_{out}=-1+\chi\gamma\left(1+\nu-\frac{1}{\sigma}+\frac{1}{\gamma}\right)=0, \tag{IA.77}$$

given the formula above for χ. Participation is now a good traded on a competitive market. Hence the first welfare theorem applies, and there are no outcome-based spillovers.

Now we apply the reasoning from Appendix D.1 to find the condition for convergence when θ is large. The condition for convergence of $\mathcal{E}_{\widetilde{\mathcal{I}}_n}$ is the same as for $\mathcal{E}_{\mathcal{I}_n}$:

$$\gamma(\theta+\chi)>1+\chi\left(\frac{\gamma}{\sigma}-1\right) \tag{IA.78}$$

$$\iff\quad \gamma\theta>1+\frac{1}{\nu+\frac{1}{\gamma}+\frac{\sigma-1}{\sigma}}\left(\frac{1}{\sigma}-\frac{1}{\gamma}-1\right). \tag{IA.79}$$

As $n\to\infty$, we have that $\widetilde{\mathcal{I}}_n\to\infty$ and $\mathcal{I}_1\to 0$, and therefore, $\mathcal{I}_1/\widetilde{\mathcal{I}}_n\to 0$.

For the high-disagreement market-based spillover we have

$$spill_{mkt}(n\to\infty)=-(\sigma-1)\cdot\mathcal{E}_w=-\frac{\sigma-1}{\gamma}-(\sigma-1)\chi\left(\frac{1}{\sigma}-\frac{1}{\gamma}\right) \tag{IA.80}$$

$$=-\frac{\sigma-1}{\gamma}\cdot\left(\frac{1+\nu}{1+\nu+1/\gamma-1/\sigma}\right). \tag{IA.81}$$

Again, Proposition 4 holds. Moreover, $|spill_{mkt}(n\to\infty)|$ is decreasing in ν since $1/\gamma<1/\sigma$. As the cost of producing infrastructure becomes steeper, the sensitivity of firm participation to firm creation is smaller, and the market-based spillover is less responsive. In the limit with $\nu \to \infty$, a fixed number of firms produces, and we are back to our baseline $spill_{mkt}(n \to \infty) = -(\sigma - 1)/\gamma$. Finally, $|spill_{mkt}(n \to \infty)|$ is increasing in σ.

D.5 Melitz (2003) Model: Participation Costs and Dixit-Stiglitz

We now introduce Dixit-Stiglitz preferences to the above model, as in Melitz (2003).

D.5.1 Setting and Equilibrium

Participating firms. Given M_e and M, profits before the infrastructure costs are unchanged from the standard model with decreasing returns to scale:

$$\pi(a) = \frac{1}{\sigma}\left(\frac{\sigma}{\sigma - 1}\right)^{1-\sigma} \cdot \mathcal{C} \cdot w^{1-\sigma} \cdot a^{\sigma-1}.$$

The equilibrium consumption is also unchanged:

$$\frac{\mathcal{C}}{L} = \left(\frac{\gamma}{\gamma - (\sigma - 1)}\right)^{\frac{1}{\sigma-1}} M^{\frac{1}{\sigma-1} - \frac{1}{\gamma}} M_e^{\frac{1}{\gamma}}.$$

The marginal firm has productivity $\underline{a}$ and spends all of its profit on infrastructure. Therefore, we have the zero-cutoff-profit condition $\Phi'(M) = \pi(\underline{a})$, which implies

$$M^{\nu + \frac{1}{\gamma} + \frac{\sigma-2}{\sigma-1}} = \frac{1}{\varphi_0}\frac{1}{\sigma}\left(\frac{\gamma - (\sigma - 1)}{\gamma}\right)^{\frac{\sigma-2}{\sigma-1}} L \cdot M_e^{\frac{1}{\gamma}},$$

where we use the fact that $\underline{a} = (M_e/M)^{1/\gamma}$. In Section D.2, we specified an exogenous set of producing firms $M = M_e^{\chi}/M_0^{\chi-1}$. This arises endogenously through our cost of infrastructure with

$$\chi = \frac{1}{\gamma}\left(\nu + \frac{1}{\gamma} + \frac{\sigma - 2}{\sigma - 1}\right)^{-1},$$

$$M_0^{1-\chi} = \left(\frac{1}{\varphi_0}\frac{1}{\sigma}\left(\frac{\gamma - (\sigma - 1)}{\gamma}\right)^{\frac{\sigma-2}{\sigma-1}} L\right)^{\left(\nu + \frac{1}{\gamma} + \frac{\sigma-2}{\sigma-1}\right)^{-1}},$$

where the exponent satisfies $\chi \leq 1$ if and only if $\nu + \frac{\sigma-2}{\sigma-1} \in (-\infty, -1/\gamma) \cup [0, \infty)$. Otherwise, all firms participate as M_e grows to infinity.

Finally, we derive the elasticity $\mathcal{E}_{\mathcal{C}}$:

$$\mathcal{E}_{\mathcal{C}} = \frac{1}{\gamma} + \chi\left(\frac{1}{\sigma - 1} - \frac{1}{\gamma}\right) = \frac{1}{\gamma} \cdot \frac{1 + \nu}{1 + \nu + 1/\gamma - 1/(\sigma - 1)} = \chi \cdot (1 + \nu).$$

Equilibrium. The equilibrium condition in the competitive equilibrium is:

$$W'(M_e) = \frac{1}{\sigma} \cdot \frac{\mathcal{C}}{M_e} \cdot \frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}.$$

Aggregate profits represent a fraction $(\sigma-1)/\gamma$ of aggregate revenue after labor costs, and aggregate infrastructure costs account for the other $(\gamma-(\sigma-1))/\gamma$. Therefore, aggregate profits represent a share $(\sigma-1)/(\sigma\gamma)$ of consumption and aggregate infrastructure costs $(\gamma-(\sigma-1))/(\sigma\gamma)$.

D.5.2 Spillovers

The market-based social value is:

$$\begin{aligned}&\frac{d}{dM_e}\left[\frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}+\frac{\sigma-1}{\sigma}\mathcal{C}+\left(\frac{\gamma-(\sigma-1)}{\sigma\gamma}\right)\mathcal{C}-\Phi(M)\right]\\ =\;&\frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}\frac{\widetilde{\mathcal{I}}'_n}{\widetilde{\mathcal{I}}_n}+\frac{1}{\sigma}\mathcal{C}'\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}-\frac{1}{\sigma}\mathcal{C}\frac{\widetilde{\mathcal{I}}_n}{\mathcal{I}_1}\frac{\mathcal{I}'_1}{\mathcal{I}_1}+\frac{\sigma-1}{\sigma}\mathcal{C}'+\left(\frac{\gamma-(\sigma-1)}{\sigma\gamma}\right)\mathcal{C}'-\Phi'(M)M\frac{1}{M}\frac{dM}{dM_e}.\end{aligned}$$

The market-based spillover is therefore:

$$spill_{mkt}(n)=\mathcal{E}_{\widetilde{\mathcal{I}}_n}-\mathcal{E}_{\mathcal{I}_1}-1+\mathcal{E}_{\mathcal{C}}+\left[(\sigma-1)\mathcal{E}_{\mathcal{C}}+\left(1-\frac{\sigma-1}{\gamma}\right)(\mathcal{E}_{\mathcal{C}}-\chi)\right]\frac{\mathcal{I}_1}{\widetilde{\mathcal{I}}_n}. \tag{IA.82}$$

The outcome-based spillover (and market-based spillover under agreement) is:

$$\begin{aligned}spill_{out}&=-1+\mathcal{E}_{\mathcal{C}}+\left[(\sigma-1)\mathcal{E}_{\mathcal{C}}+\left(1-\frac{\sigma-1}{\gamma}\right)(\mathcal{E}_{\mathcal{C}}-\chi)\right]\frac{\gamma}{\sigma-1} &\text{(IA.83)}\\ &=\frac{1}{\sigma-1}\cdot\frac{1+\nu}{1+\nu+1/\gamma-1/(\sigma-1)}. &\text{(IA.84)}\end{aligned}$$

Now we apply the reasoning from Section D.1 to find the condition for convergence when θ is large. The condition for convergence of $\mathcal{E}_{\widetilde{\mathcal{I}}_n}$ is the same as for $\mathcal{E}_{\mathcal{I}_n}$:

$$\begin{aligned}&\gamma(\theta+\chi)>1+\chi\left(\frac{\gamma}{\sigma-1}-1\right) &\text{(IA.85)}\\ \iff\quad&\gamma\theta>1+\frac{1}{\nu+\frac{1}{\gamma}+\frac{\sigma-2}{\sigma-1}}\left(\frac{1}{\sigma-1}-\frac{1}{\gamma}-1\right). &\text{(IA.86)}\end{aligned}$$

As $n\to\infty$, we have $\widetilde{\mathcal{I}}_n\to\infty$ and $\mathcal{I}_1\to 0$, and therefore $\mathcal{I}_1/\widetilde{\mathcal{I}}_n\to 0$.

For the high-disagreement market-based spillover we have

$$spill_{mkt}(n\to\infty)=-(\sigma-1)\cdot\mathcal{E}_w+\mathcal{E}_{\mathcal{C}}=-\frac{\sigma-2}{\gamma}\cdot\left(\frac{1+\nu}{1+\nu+1/\gamma-1/(\sigma-1)}\right). \tag{IA.87}$$

When $\nu\to\infty$, there is a fixed supply of infrastructure and thus a fixed number of firms, which implies:

$$\begin{aligned}spill_{out}&=\frac{1}{\sigma-1},\\ spill_{mkt}(n\to\infty)&=-\frac{\sigma-2}{\gamma}.\end{aligned}$$

E Data Appendix

E.1 Data Construction Details

Bubbles. Following Greenwood, Shleifer, and You (2018), we identify bubbles as episodes in which stock prices of an industry have increased over 100% in terms of both raw and net-of-market returns over the previous two years, followed by a decrease in absolute terms of 40% or more. Industries are classified according to the Fama-French 49-industry scheme, and the data begin in 1928.

Value of Innovation. We use the stock market value of patents at the patent level and at the firm level directly from Kogan et al. (2017), as well as the number of citations that accrue to a patent.[42]

Compustat Segments. We merge the Compustat funda file with the Compustat segments file. We estimate the number of segments with different industry codes. The Compustat segments file provides both a six- and a four-digit industry code, which gives two measures of the number of different types of industries within a public firm.

Value of Spillovers. We obtain information on the quantity of competition spillovers (variable spillsic) as well as technological spillovers (variable spilltec) from the replication files in Bloom, Schankerman, and Van Reenen (2013). The exposure to spillovers from product market, *spillsic*, is defined as the correlation of the sales across two firms' Compustat segments. If we consider the vector of average sales share across each industry for a given firm i, $\mathbf{S}_i$, product market proximity between firm i and j is defined by the uncentered correlation: $\text{SIC}_{ij} = \mathbf{S}_i\mathbf{S}_j'/\left(\sqrt{\mathbf{S}_i\mathbf{S}_i'}\sqrt{\mathbf{S}_j\mathbf{S}_j'}\right)$. The product market spillover is the average stock of R&D that is in the product market proximity of firm i:

$$spillsic_i = \sum_{j\neq i} \text{SIC}_{ij} G_j,$$

where G_j is the stock of R&D for firm j. The exposure to knowledge spillovers is constructed the same way, where we define for firm i a vector of share of patents across technology classes from the USPTO as $\mathbf{T}_i$. The uncentered correlation of technology between firm i and j is: $\text{TECH}_{ij} = \mathbf{T}_i\mathbf{T}_j'/\left(\sqrt{\mathbf{T}_i\mathbf{T}_i'}\sqrt{\mathbf{T}_j\mathbf{T}_j'}\right)$. The product market spillover is the average stock of R&D that is in the product market proximity of firm i:

$$spilltech_i = \sum_{j\neq i} \text{TECH}_{ij} G_j.$$

To look at the effect of spillovers, we use sales item from Compustat funda file and Tobin's q (market-to-book ratio) from the CRSP-Compustat merged file.

[42] We thank Dimitris Papanikolaou for graciously sharing his data with us.

E.2 Supplementary Tables

Table IA.1
Innovation in Times of Bubbles

	Patents (#)	Log Patents (#)			
	(1)	(2)	(3)	(4)	(5)
Bubble	1.385**	0.148***	0.154**	0.169***	0.178**
	(0.578)	(0.056)	(0.066)	(0.063)	(0.075)
Lagged Citations	0.982***				
	(0.026)				
Lagged Log Citations		0.824***	0.795***	0.827***	0.799***
		(0.008)	(0.008)	(0.007)	(0.008)
Fixed Effects	C	–	C	Y	C, Y
Observations	106,176	106,278	106,176	106,278	106,176
R^2	0.91	0.67	0.68	0.68	0.68

Note: Table IA.1 presents panel regressions of the quantity of innovation, measured by the number of patents issued at the USPTO three-digit class level, on a dummy from Greenwood, Shleifer, and You (2018) that captures whether the firm is in an industry that is in a bubble or not. We control for the lagged number of patents for column one and lagged logarithm for columns two to five. Depending on the specification, we include fixed effects for the patent class level C and patent grant-year Y. Standard errors clustered at the grant-year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

Table IA.2
Private Value of Innovation in Bubbles

	Patent Level			Firm Level		
	(1)	(2)	(3)	(4)	(5)	(6)
Bubble Dummy	0.317***	0.289***	0.277***	0.514***	0.429***	0.431***
	(0.094)	(0.090)	(0.083)	(0.114)	(0.123)	(0.080)
Log Market Cap			0.543***			0.625***
(lagged)			(0.026)			(0.020)
Fixed Effects	Y, F	Y, F	Y, F	Y, F	Y, F	Y, F
Interaction	NA	Ind x Cite	Ind x Cite	NA	Ind x Cite	Ind x Cite
Observations	1,171,806	1,118,675	1,116,740	47,886	47,886	47,484
R^2	0.68	0.69	0.74	0.89	0.94	0.96

Note: Table IA.2 presents panel regressions of the value of innovation, as measured in Kogan et al. (2017) at the patent firm levels, on a dummy from Greenwood, Shleifer, and You (2018) that captures whether the firm is in an industry that is in a bubble or not. We control for the forward-looking number of citations generated by a patent (or firm) from Kogan et al. (2017) and the lagged market capitalization of the firm. We include firm fixed effects F and patent grant year fixed effects Y. Depending on the specification, we also use industry fixed effects (from the Fama-French 49-industry classification) interacted with the log number of forward-looking citations to allow for different slopes in the relation between private valuation and the patent quality across industries. Standard errors clustered at the grant-year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

Table IA.3
Diversity and Private Value of Innovation in Bubbles

	Patent Level			Firm Level		
	(1)	(2)	(3)	(4)	(5)	(6)
Bubble x Segments	-0.562^{***}	-0.550^{***}	-0.488^{***}	-0.388^{**}	-0.370^{***}	-0.295^{***}
(NAICS 6 digits)	(0.178)	(0.175)	(0.164)	(0.155)	(0.094)	(0.072)
Bubble	1.456^{***}	1.425^{***}	1.329^{***}	1.459^{***}	1.389^{***}	1.167^{***}
	(0.232)	(0.227)	(0.235)	(0.317)	(0.342)	(0.289)
Segments	0.122	0.122	0.112	0.010	-0.031	-0.026
(NAICS 6 digits)	(0.096)	(0.095)	(0.101)	(0.044)	(0.048)	(0.041)
Log Citations		0.049^{***}	0.047^{***}		0.047^{***}	0.047^{***}
(forward looking)		(0.010)	(0.009)		(0.009)	(0.009)
Log Market Cap			0.156^{***}			0.287^{***}
(lagged)			(0.042)			(0.046)
Fixed Effects	Y, F	Y, F	Y, F	Y, F	Y, F	Y, F
Observations	180,636	180,636	177,911	10,426	10,426	10,256
R^2	0.71	0.71	0.72	0.88	0.93	0.94

Note: Table IA.3 presents panel regressions of the value of innovation, as measured in Kogan et al. (2017) at the patent and firm levels, on a bubble dummy interacted with the number of industries spanned by the different segments of the parent firm. The bubble dummy is from Greenwood, Shleifer, and You (2018) and captures whether the firm is in an industry that is in a bubble or not. Compustat segments are measured at the six-digit NAICS code level from the Compustat segments file. We control for the forward-looking number of citations generated by a patent (or firm) from Kogan et al. (2017) and the lagged market capitalization of the firm. We also include fixed effects for firm F and patent grant year Y. Standard errors clustered at the grant-year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

Table IA.4
Forward Citations and Patent Market Values

	Table II from Kogan et al. (2017)					Firm-Year Fixed Effects		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Log Citations	0.174***	0.099***	0.054***	0.013***	0.004***	0.023***	0.019***	0.012***
	(0.017)	(0.010)	(0.005)	(0.001)	(0.001)	(0.004)	(0.003)	(0.002)
Controls								
Firm Size	—	✓	✓	✓	✓	—	✓	✓
Volatility	—	—	✓	✓	—	—	—	✓
Fixed Effects	CxY	CxY	CxY	CxY, F	CxY, FxY	Y, F	Y, F	Y, F
Observations	1,801,301	1,801,301	1,801,301	1,801,301	1,801,301	1,801,301	1,801,301	1,801,301
R^2	0.20	0.71	0.79	0.93	0.95	0.82	0.91	0.92

Note: Table IA.4 presents panel regressions of the stock market value of innovation on the logarithm of the number of citations received by a patent until the end of the sample in 2010. Depending on the specification, we control for the logarithm of firm size and volatility. Depending on the specification, we include patent grant times year fixed effects, CxY, year fixed effects F, and firm fixed effects Y. Columns (1) to (5) reproduce Table II from Kogan et al. (2017), while columns (6) to (8) only include year and firm fixed effects to be comparable with the tables above. Standard errors clustered at the grant-year level are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.